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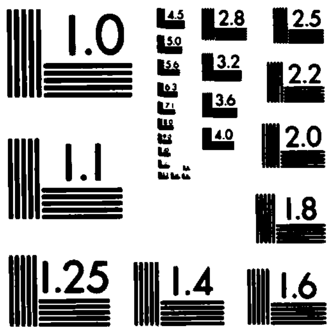

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FINAL REPORT

COHERENT ELECTROMAGNETIC WAVE PROPAGATION AND SCATTERING IN RANDOM MEDIA

by

V. K. Varadan* and V. V. Varadan*

Wave Propagation Group
Department of Engineering & Mechanics and
Atmospheric Sciences Program
Ohio State University, Columbus, Ohio 43210

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*Present address: Department of Engineering Science and Mechanics, Pennsylvania State University, University Park, PA 16802

SUMMARY

In this report the effect of multiple scattering on the coherent wave propatation in discrete random media has been investigated. The media is modelled as a random distribution of spherical and non-spherical scatterers. The first and second order probability distribution functions are specified and a self-consistent T-matrix approach together with Lax's guasi-crystalline approximation is used to derive dispersion equations whose singular solutions yield the complex propagation constant of the "effective" medium.

Our formalism is well suited for numerical computations for wavelengths comparable to scatter size and high volume concentration of scatterers; to our knowledge, it is the only method that provides reliable numerical results for the attenuation and phase velocity of the coherent wave. The quasi-crystalline approximation (QCA) is found to be applicable for all concentration values from low to high. The QCA that is used to truncate the hierarchy of equations during the configurational averaging procedure requires knowledge of the two particle joint probability distribution function.

Our model of the random system is that the sphere circumscribing the scatterers cannot interpenetrate. In the statistical mechanics literature this is synonymous with ensemble of 'hard spheres.' For such a model, several forms of the pair-correlation function can be obtained from the statistical mechanics of simple liquids by using several theories and calculations such as the Hypernetted-chain Equation (HNC), the Percus-Yevick Approximation (P-YA), the Self-Consistent Approximation (SCA), Monte Carlo calculations, etc. For the hard sphere model, temperature and other thermodynamic quantities do not appear in the final form of the correlation functions and hence are equally valid for our system. Five

forms of the pair correlation function were considered:

(a) Well stirred approximation (WSA). This is the simplest and incorporates only the whole correction. In the Rayleigh limit, for $c > \cdot 125$, this model fails. But with increasing frequency, it is good for higher values of c. At $ka = \omega a/c \sim 5.0$, there is no difference between this and other models suggesting that correlations are unimportant at high values of the wavenumber.

(b) The Percus-Yevick approximation can be used to obtain a semi-analytical form of the pair correlation function. It is good for $c \lesssim 0.35$.

(c) Virial Expansion in powers of the number density can also be conveniently used since analytical expansions are available. Obviously it works only for $c < \cdot 1$.

(d) The Matern model is an analytical model for the pair correlation function that is also convenient to use for $c \lesssim \cdot 125$. The Matern model is found to be superior to the WSA for $c \leq 0.125$.

(e) The self-consistent approximation to the pair correlation function which is based on the Percus-Yevick model and the Hyper-netted Chain approximation is the one that has proved most successful in our computations. It provides reliable results for a wide range of scatterer concentrations.

All of the above forms were tested for several values of c and ka. Recently our calculations were compared with the experimental findings of Professor A. Ishimaru for a distribution of latex spheres in water. The agreement is excellent for all cases, see our paper Nos: 1, 3 and 4. In paper Nos: 4 and 5, we have introduced the concept of an average T-matrix to include a size distribution for scatterers using the Gaussian distribution function. The question may be raised as to the radius of the excluded volume surrounding each scatterer. If we use $4\bar{a}$, the diameter of the largest size sphere of the distribution, although we would be correct in the integration on the allowed position for each scatterer, we would be excluding an unreasonable volume, thus limiting the volume fraction of scatterers. It seems more reasonable to take the diameter of the mean sphere, $2\bar{a}$ as the radius of the excluded spherical volume. On the average, this would be applicable to most scatterers and allow us to consider higher volume fractions. If the volume fraction is low enough, whether the radius of the excluded sphere is $2\bar{a}$ or $4\bar{a}$ will not pose a problem, see our paper No: 6 for more details. In Paper No. 5, we have compared Keller's and Twersky's multiple scattering approaches with our formalism. Our formalism is in exact agreement with Twersky's approach. However, it is better suited for numerical computations. Recently, we have also performed calculations on coated dielectric spheres in free space or in another dielectric. Numerical results are obtained for the complex average dielectric constant, coherent attenuation and phase velocity as a function of concentration ($0 \leq c \leq 0.42$). A plot of coherent attenuation vs c for coated spheres is attached with this report, see Fig. 1. A T-matrix program has also been developed to obtain the T-matrix of coated irregular arbitrary shaped bodies which can be implemented in our multiple scattering approach to study wave propagation through snow, aerosol and various kinds of debris. Numerical results of irregular arbitrary shaped bodies will be forthcoming.

Recently, we had extended our formalism to study high frequency propagation of waves in random media. We have compared our results with those of experimental findings obtained by Ishimaru for ka as high as 83.596 and the agreement is found to be excellent, see Fig. 2. We are confident that we have a well founded formalism and a sound numerical technique along with our T-matrix and finite element (uni-moment) approaches for the study of wave propagation in random media.

Various publications resulting from our investigations are enclosed.

SCIENTIFIC PERSONNEL SUPPORTED BY THIS PROJECT

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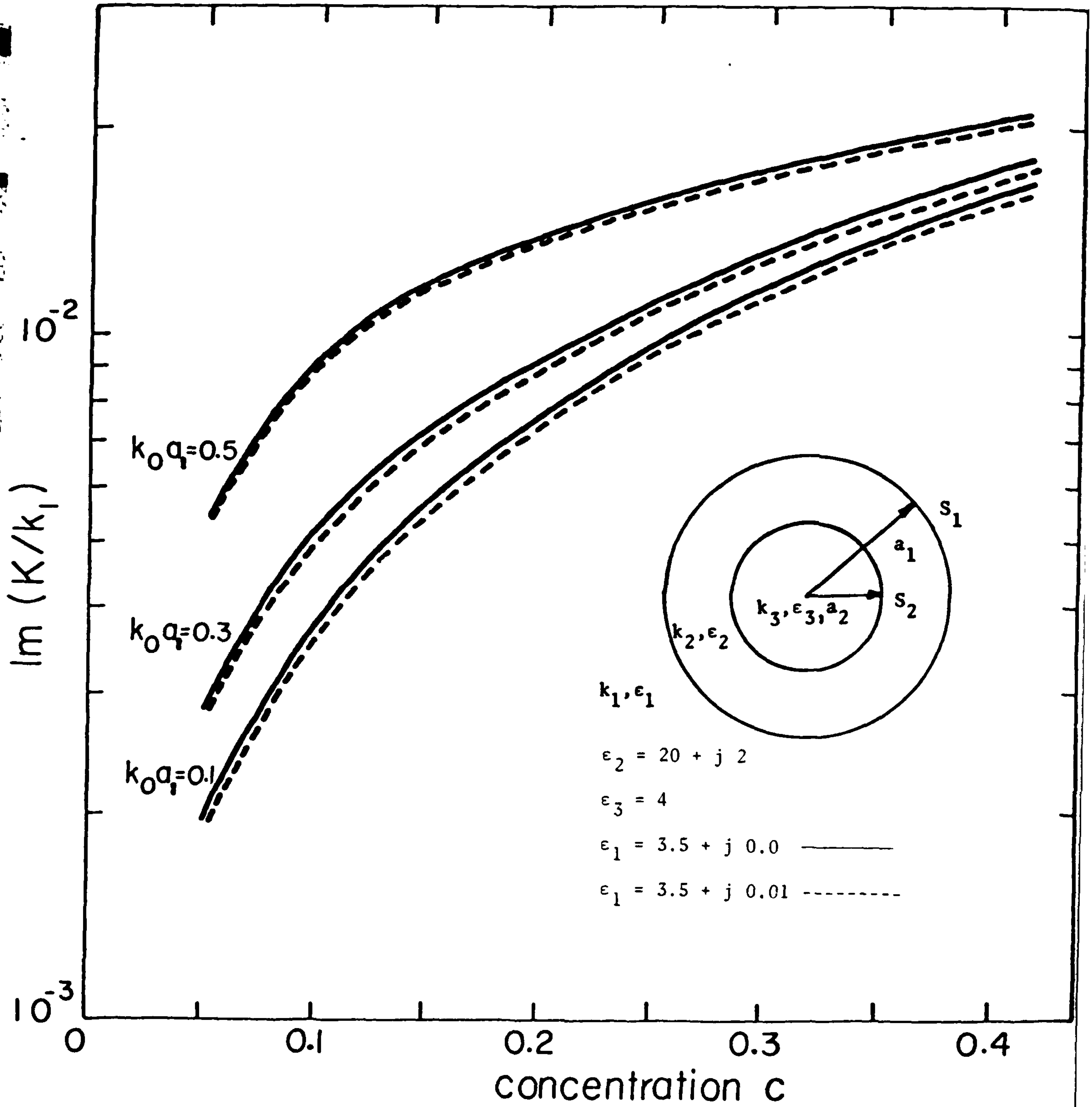

Fig. 1. Attenuation Im(K/k_1) as a function of concentration c of coated spheres (a_1/a_2 = 1.17) for three values of the free space wavenumber k_0a_1

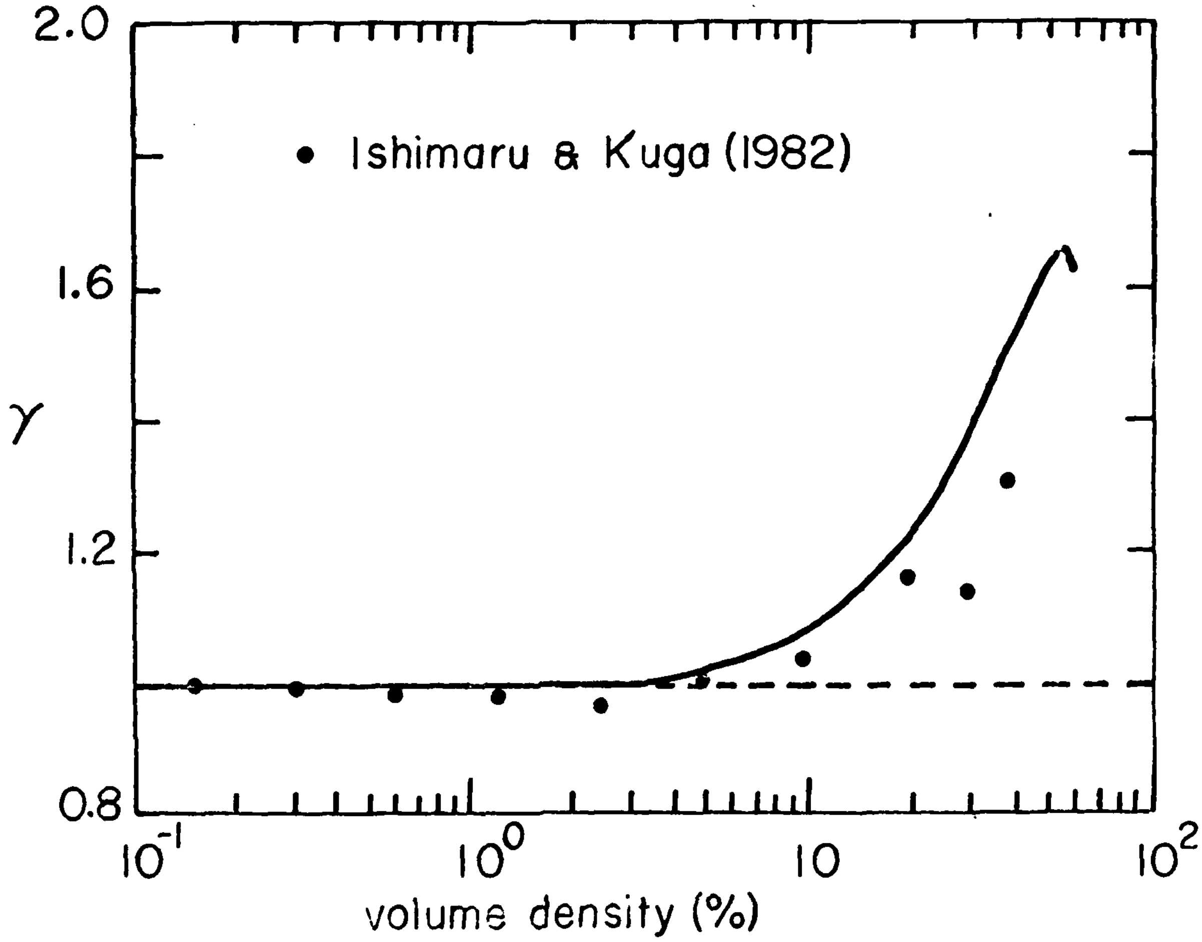

Fig. 2. Plot of $\gamma = 2 \text{ Im } K/n_0\sigma_t$ versus concentration where n_0 is the number density and σ_t is the extinction cross section of a single sphere with ka = 83.596.

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Coherent wave attenuation by a random distribution of particles

V. N. Bringi[1]

Wave Propagation Group, Boyd Laboratory, and Department of Electrical Engineering
Ohio State University, Columbus, Ohio 43210

V. V. Varadan and V. K. Varadan

Wave Propagation Group, Boyd Laboratory, Department of Engineering Mechanics, and Atmospheric Sciences Program
Ohio State University, Columbus, Ohio 43210

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Coherent electromagnetic wave propagation in an infinite medium composed of a random distribution of identical, finite scatterers is studied. A self-consistent multiple scattering theory using the T matrix of a single scatterer and a suitable averaging technique is employed. The statistical nature of the position of scatterers is accounted for by ensemble averaging. This results in a hierarchy of equations relating the different orders of correlations between the scatterers. Lax's quasi-crystalline approximation is used to truncate the hierachy enabling passage to a homogeneous continuum whose bulk propagation characteristics such as phase velocity and coherent wave attenuation can then be studied. Three models for the pair correlation function are considered. The Matern model and the well-stirred approximation are good only for sparse concentrations, while the Percus-Yevick approximation is good for a wider range of concentration. The results obtained using these models are compared with the available experimental results for dielectric scatterers embedded in a host dielectric medium. Practical applications of this study include artificial dielectric (composites) and electromagnetic wave propagation through hydrometeors, dust, vegetation, etc.

1. INTRODUCTION

We consider the propagation of plane coherent electromagnetic waves in an infinite medium containing identical, lossless, randomly distributed particles. Our aim is to characterize the random medium by an effective complex wave number K which would be a function of the particle concentration, the electrical size, and the statistical description of the random positions of the scatterers. The imaginary part of K describes the coherent attenuation which is due to multiple scattering only when the particles themselves are assumed to be lossless. The understanding of the behavior of Im (K) as a function of particle concentration (c) and/or frequency (ka) is very important in many practical applications, including wave propagation in the atmosphere and oceans and whenever random distributions of scatterers influence electromagnetic wave behavior.

The theoretical formulation presented here closely follows the procedure described by *Varadan et al.* [1979, 1982], *Varadan and Varadan* [1980], and *Varadan* [1980]. This approach is based on a self-consistent multiple scattering theory and relies on the T matrix [*Waterman*, 1971] which relates the field scattered by a particle to an arbitrary exciting field. The statistical description of the random positions of the scatterers is used to define a configurational average which results in a hierarchy of equations relating the different orders of correlations between the scatterers. *Lax's* [1952] quasi-crystalline approximation (QCA) is used to truncate the hierarchy which results in the usual 'hole correction' integrals. Following *Twersky* [1977, 1978*a*, *b*], a radially symmetric pair correlation function is introduced, and approximate models are chosen from *Talbot and Willis* [1980]. The 'well-stirred' approximation (WSA) which was used previously by *Varadan et al.* [1979] and *Bringi et al.* [1981] assumes no corre-

[1] Now at Colorado State University, Fort Collins, Colorado 80523.

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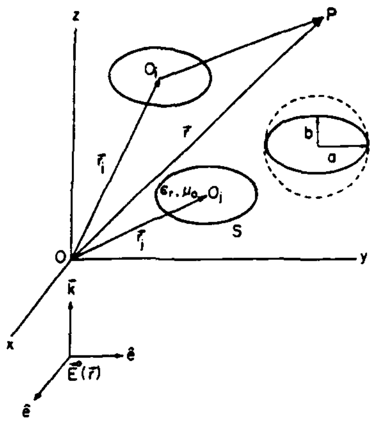

Fig. 1. Geometry of randomly distributed and aligned scatterers.

lation between the particles except that they should not interpenetrate. In particular, the WSA gives unphysical results for $c \geq 0.125$ at the Rayleigh or low-frequency limit.

In this paper we consider two other pair correlation functions, viz., (1) the *Matern* [1960] model and (2) the Percus-Yevick [*Percus and Yevick*, 1958] model for a classical system of hard spheres. Computations of Im (K) are presented for dielectric scatterers imbedded in a host dielectric medium, using the above three models as a function of frequency and concentration. We compare our solutions to some recent optical propagation experiments conducted by Ishimaru (A. Ishimaru, personal communication, 1981). Sample computations are also presented comparing the WSA and the single scattering approximation for a rain medium.

2. FORMULATION OF THE PROBLEM

Consider N identical, finite dielectric scatterers that are randomly distributed either in free space or in a host dielectric medium. The scatterers are homogeneous with a relative dielectric constant of ε_r, their centers being denoted by 0_1, 0_2, 0_3, $\cdots$, 0_N (see Figure 1). They are assumed to be bodies of revolution with symmetry axis parallel to the z direction. A monochromatic, plane, coherent electromagnetic wave is assumed to propagate along the symmetry axis of the scatterers so as to satisfy the condition that the effective medium be isotropic and polarization insensitive. The time dependence of the incident field and hence the fields scattered by the individual scatterers is of the form $\exp(-j\omega t)$ and is suppressed in the equations that follow. Even though the theory presented here is valid for spheroidal scatterers [*Varadan et al.*, 1982], we present computations only for spherical scatterers in order to compare our results with available experiments.

Let $\mathbf{E}^0(\mathbf{r})$ be the electric field arising from the incident plane wave and $\mathbf{E}_i^S(\mathbf{r})$ the field scattered by the ith scatterer. Both these fields satisfy the vector Helmholtz equation. The total field at any point outside the scatterers is given by the sum of the incident field and the fields scattered by all the scatterers, which can be written as

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^0(\mathbf{r}) + \sum_{i=1}^{N} \mathbf{E}_i^S(\boldsymbol{\rho}_i) \qquad \boldsymbol{\rho}_i = \mathbf{r} - \mathbf{r}_i \tag{1}$$

where $\mathbf{E}_i^S(\boldsymbol{\rho}_i)$ is the field scattered by the ith scatterer at the observation point $\mathbf{r}$. However, the field that excites the ith scatterer is the incident field $\mathbf{E}^0$ plus the fields scattered from all other scatterers except the ith. The term exciting field $\mathbf{E}^e$ is used to distinguish between the field actually incident on a scatterer and the external incident $\mathbf{E}^0$ produced by a source at infinity. Thus at a point $\mathbf{r}$ in the vicinity of the ith scatterer, we write

$$\mathbf{E}_i^e(\mathbf{r}) = \mathbf{E}^0(\mathbf{r}) + \sum_{j \neq i}^{N} \mathbf{E}_j^S(\boldsymbol{\rho}_j) \qquad a \leq |\rho_j| < 2a \tag{2}$$

where a is a typical dimension of the scatterer.

The exciting and scattered fields for each scatterer can be expanded in terms of vector spherical functionswith respect to an origin at the center of that scatterer:

$$\mathbf{E}_i^e(\mathbf{r}) = \sum_{\tau=1}^{2} \sum_{l=1}^{\infty} \sum_{m=0}^{l} \sum_{\sigma=1}^{2} b^i_{\tau l m \sigma} \operatorname{Re} \boldsymbol{\psi}_{\tau l m \sigma}(\boldsymbol{\rho}_i)$$
$$= \sum_{\tau n} b^i_{\tau n} \operatorname{Re} \boldsymbol{\psi}^i_{\tau n} \tag{3}$$

and

$$\mathbf{E}_i^S(\mathbf{r}) = \sum_{\tau n} B^i_{\tau n} \boldsymbol{\psi}^i_{\tau n} \tag{4}$$

where n represents a combined index notation for the set $[l, m, \sigma]$.

The vector spherical functions are defined as

$$\boldsymbol{\psi}_{1lm\sigma}(\mathbf{r}) = \nabla \times [\mathbf{r} h_l(kr)] Y_{lm\sigma}(\theta, \phi) \tag{5}$$

$$\boldsymbol{\psi}_{2lm\sigma}(\mathbf{r}) = (1/k) \nabla \times \boldsymbol{\psi}_{1lm\sigma}(\mathbf{r}) \tag{6}$$

In (3–6), k is the wave number, h_l is the spherical Hankel function of the first kind, and the $Y_{lm\sigma}(\theta, \phi)$ are the normalized spherical harmonics defined with real angular functions. In (3) the exciting field is expanded in terms of the regular (Re) basis set (Re $\psi^i_{\tau n}$) obtained by replacing h_n in (5) and (6) by j_n, the spherical Bessel functions of the first kind. Thus the choice of the basis set in (4) satisfies the radiation condition at infinity for the scattered field, while the choice in (3) satisfies the regular behavior of the exciting field in the region $a < |\rho_i| < 2a$. The superscript i on the basis functions refer to expansions with respect to 0_i, and $b^i_{\tau n}$ and $B^i_{\tau n}$ are the unknown exciting and scattered field coefficients, respectively. We also expand the incident field in terms of vector spherical functions:

$$\mathbf{E}^0(\mathbf{r}) = e^{ik\hat{z}\cdot\mathbf{r}_i} \sum_{\tau n} a_{\tau n} \operatorname{Re} \psi^i_{\tau n}(\boldsymbol{\rho}_i) \tag{7}$$

where the $a_{\tau n}$ are the known incident field coefficients.

The unknown coefficients $b^i_{\tau n}$ can be related to $B^i_{\tau n}$ by means of any convenient scattering operator; in this case we employ the T matrix as defined by *Waterman* [1971]:

$$B^i_{\tau n} = \sum_{\tau' n'} T^i_{\tau n, \tau' n'} b^i_{\tau' n'} \tag{8}$$

Substituting (3), (4), and (7) in (2), we obtain

$$\sum_{\tau n} b^i_{\tau n} \operatorname{Re} \psi^i_{\tau n} = e^{ik\hat{z}\cdot\mathbf{r}_i} \sum_{\tau n} a_{\tau n} \operatorname{Re} \psi^i_{\tau n} + \sum_{j \neq i}^{N} \sum_{\tau n} B^j_{\tau n} \psi^j_{\tau n} \tag{9}$$

Since the field quantities are expanded with respect to centers of each scatterer, we obtain (9) with basis functions expanded with respect to ith and jth centers. In order to express them with respect to a common origin 0_i, we employ the translation addition theorems for the vector spherical functions (see, for example, *Boström*, [1980]) which can be written in a compact form as follows:

$$\begin{aligned} \psi_{\tau n}(\boldsymbol{\rho}_j) &= \sum_{\tau' n'} \sigma_{\tau n, \tau' n'}(\boldsymbol{\rho}_{ij}) \operatorname{Re} \psi_{\tau' n'}(\boldsymbol{\rho}_i) \qquad |\boldsymbol{\rho}_{ij}| > |\boldsymbol{\rho}_i| \\ \psi_{\tau n}(\boldsymbol{\rho}_j) &= \sum_{\tau' n'} R_{\tau n, \tau' n'}(\boldsymbol{\rho}_{ij}) \cdot \psi_{\tau' n'}(\boldsymbol{\rho}_i) \qquad |\boldsymbol{\rho}_{ij}| < |\boldsymbol{\rho}_i| \end{aligned} \tag{10}$$

where $\boldsymbol{\rho}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ is the vector connecting 0_j to 0_i, $\sigma_{\tau n, \tau' n'}$ is the translation matrix for the vector functions, and $R_{\tau n, \tau' n'}$ is a matrix with spherical Hankel functions in $\sigma_{\tau n \tau' n'}$ replaced by spherical Bessel functions.

Employing (8) and (10) in (9) and using the orthogonality of the vector spherical functions, we obtain the following set of algebraic equations for the exciting field coefficients $b^i_{\tau n}$:

$$b^i_{\tau n} = e^{ik\hat{z}\cdot\mathbf{r}_i} a_{\tau n} + \sum_{j \neq i}^{N} \sum_{\tau' n'} \sum_{\tau'' n''} \sigma_{\tau' n', \tau n}(\boldsymbol{\rho}_{ij}) \cdot T^j_{\tau' n', \tau'' n''} b^j_{\tau'' n''} \tag{11}$$

From (11) it can be seen that the exciting field coefficients of the ith scatterer explicitly depend on the position and orientation of the other scatterers. In this paper we consider a random distribution of spherical scatterers and the case when both $N \to \infty$ and the total volume accessible to the scatterers $V \to \infty$ such that $N/V = n_0$ is a finite number density. For such distributions a configurational average of (11) can be made over the positions of all the scatterers [see *Varadan et al.*, 1982]. The quasi-crystalline approximation [*Lax*, 1952] can then be invoked to arrive at an equation for the configurational average $\langle b^i_{\tau n} \rangle_i$ of the exciting field coefficients with one scatterer fixed:

$$\langle b^i_{\tau n} \rangle_i = e^{ik\hat{z}\cdot\mathbf{r}_i} a_{\tau n} + (N-1) \sum_{\tau' n'} \sum_{\tau'' n''} T_{\tau' n', \tau'' n''} \cdot \int_V p(\mathbf{r}_j | \mathbf{r}_i) \sigma_{\tau' n', \tau n}(\boldsymbol{\rho}_{ij}) \langle b^j_{\tau'' n''} \rangle_j \, d\mathbf{r}_j \tag{12}$$

where $p(\mathbf{r}_j | \mathbf{r}_i)$ is the two-particle conditional probability density.

We now assume that the average or coherent field propagates in a medium with an effective complex wave number $\mathbf{K} = (K_1 + iK_2)\hat{z}$ in the direction of the original incident field:

$$\langle \mathbf{E}^e_i(\mathbf{r}) \rangle_i = A \exp(iK\hat{z} \cdot \mathbf{r}) \tag{13}$$

where A is the amplitude of the coherent wave. Thus the average exciting field coefficients may be expressed as

$$\langle b^i_{\tau l m \sigma} \rangle_i = \exp(iK\hat{z} \cdot \mathbf{r}_i) Y_{\tau l m \sigma} \delta_{m1} [\delta_{\tau 1}\delta_{\sigma 2} + \delta_{\tau 2}\delta_{\sigma 1}] \tag{14}$$

where the Kronecker deltas in (14) indicate that only the azimuthal index $m = 1$ contributes while those in square brackets indicate that the effective medium is isotropic.

Equation (14) is now substituted in (12), and the conditional probability $p(\mathbf{r}_j | \mathbf{r}_i)$ is expressed in the form

$$\begin{aligned} p(\mathbf{r}_j | \mathbf{r}_i) &= \frac{1}{V(1-c)} g(x) \qquad x > 1 \\ p(\mathbf{r}_j | \mathbf{r}_i) &= 0 \qquad x < 1 \end{aligned} \tag{15}$$

where $x = \rho_{ij}/2a$. The function $g(x)$ is termed the

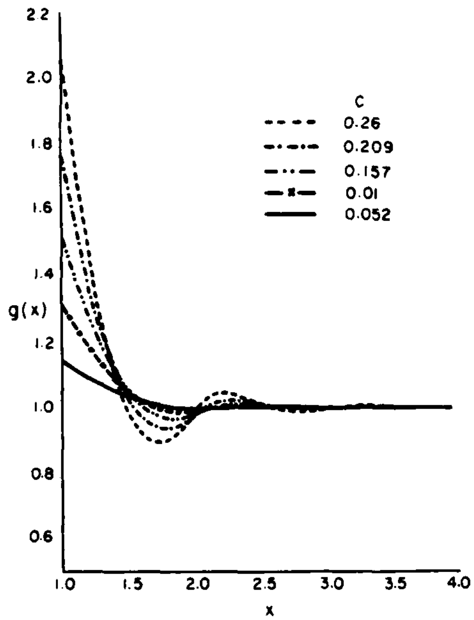

Fig. 2. The Percus-Yevick pair correlation function for hard spheres.

pair correlation function, which is assumed to be spherically symmetric. Equation (12) can now be simplified by making use of certain symmetry properties of the T matrix [*Waterman*, 1971] and by invoking the extinction theorem [*Twersky*, 1977] to yield the following set of equations for the unknown coefficients $Y_{\tau l m \sigma}$:

$$
\begin{aligned}
{}_{\sigma}\delta_{m1}[\delta_{\tau 1}\delta_{\sigma 2} + \delta_{\tau 2}\delta_{\sigma 1}]
&= \frac{N-1}{V} \sum_{\tau' l' \sigma'} \sum_{\tau'' l'' \sigma''} T_{\tau' l' 1 \sigma', \tau'' l'' 1 \sigma''} \\
&\quad \cdot \left\{ \sum_{\lambda = |l - l'|}^{l + l'} I(K, k, c, \lambda) Y_{\tau'' l'' 1 \sigma''} D^{(\lambda)}_{\tau' l' 1 \sigma', \tau l 1 \sigma} \right\}
\end{aligned}
\tag{16}
$$

where

$$
\begin{aligned}
I(K, k, c, \lambda) &= \frac{6c}{(ka)^2 - (Ka)^2} [2ka j_\lambda(2Ka) h'_\lambda(2ka) - 2Ka h_\lambda(2ka) j'_\lambda(2Ka)] \\
&\quad + 24c \int_1^{\infty} x^2 [g(x) - 1] h_\lambda(2kax) j_\lambda(2Kax)\, dx
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
D_{\tau' l' 1 \sigma', \tau l 1 \sigma}(\lambda) &= i^{l - l' + \lambda} \left[\frac{(2\lambda + 1)(2l + 1)}{2l(l + 1)} \right] \\
&\quad \cdot \left[\frac{l'(l' + 1)}{l(l + 1)} \right]^{1/2} \begin{bmatrix} l' & l & \lambda \\ 1 & -1 & 0 \end{bmatrix} \left\{ \begin{bmatrix} l' & l & \lambda \\ 0 & 0 & 0 \end{bmatrix} \right. \\
&\quad \cdot [l'(l' + 1) + l(l + 1) - \lambda(\lambda + 1)] \delta_{\tau\tau'} \delta_{\sigma\sigma'} \\
&\quad + i \begin{bmatrix} l' & l & \lambda - 1 \\ 0 & 0 & 0 \end{bmatrix} \{[\lambda^2 - (l' - l)^2][(l' + l + 1)^2 - \lambda^2]\}^{1/2} \\
&\quad \left. \cdot (1 - \delta_{\tau\tau'})(\delta_{\sigma' 1}\delta_{\sigma 2} - \delta_{\sigma' 2}\delta_{\sigma 1}) \right\}
\end{aligned}
\tag{18}
$$

In (17), $c = \frac{4}{3}\pi a^3 n_0$ is the fractional concentration by volume, and in (18),

$$
\begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{bmatrix}
$$

is the Wigner 3-j symbol. If the integral I in (17) can be evaluated for suitable models of the pair correlation function, then (16) constitutes a set of homogeneous, algebraic equations with $Y_{\tau l m \sigma}$ as the unknowns. For a nontrivial solution, the determinant of the coefficient matrix must vanish. This yields the dispersion equation for the effective, complex wave number K. In the Rayleigh or low-frequency limit, analytical solutions for K can be found while at higher frequencies the solutions can be generated computationally. The real part K_1 is the phase constant while the imaginary part is the attenuation constant of the effective medium.

If the concentration is sparse, $c \to 0$, then $g(x)$ assumes the form of a step function at $x = 1$. This behavior of $g(x)$ is termed the well-stirred approxi-

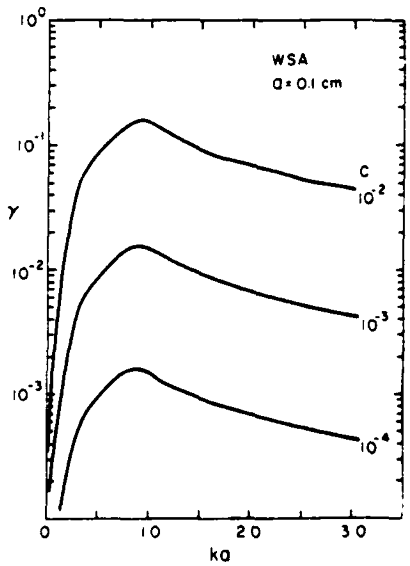

Fig. 3. The coherent attenuation constant γ versus ka for $\varepsilon_r = \varepsilon_r(\lambda)$ using the WSA.

mation (WSA) by *Talbot and Willis* [1980] and has been used previously by *Fikioris and Waterman* [1964]. *Talbot and Willis* [1980] also discuss a number of other pair correlation functions suitable for use at higher concentrations, viz., a model due to *Matern* [1960] and the model obtained by solution of the Percus-Yevick [*Percus and Yevick*, 1958] integral equation for a classical system of hard spheres [*Wertheim*, 1963]. Other forms for $g(x)$ can be used based on the statistical mechanical theory applied toward the study of dense gases and liquids [*Twersky*, 1978*b*]. In this study, however, we show sample computations using the $g(x)$ corresponding to the WSA, the Matern model which is analytic, and the Percus-Yevick approximation (P-YA) based on tabulated values of $g(x)$ given by *Throop and Bearman* [1965]. The behavior of $g(x)$ versus x is shown in Figure 2 for various values of concentration.

3. COMPUTATIONS

In many practical applications it is of interest to determine the concentration levels above which multiple scattering effects must be accounted for in a rigorous manner. At very low concentrations, $c \to 0$, the particles can be considered as essentially decorrelated so that the so-called 'single scattering' approximation (SSA) can be used leading to the following expression for Im K:

$$\mathrm{Im}\,(K/k) = \tfrac{3}{8} c Q_{ext}/(ka) \tag{19}$$

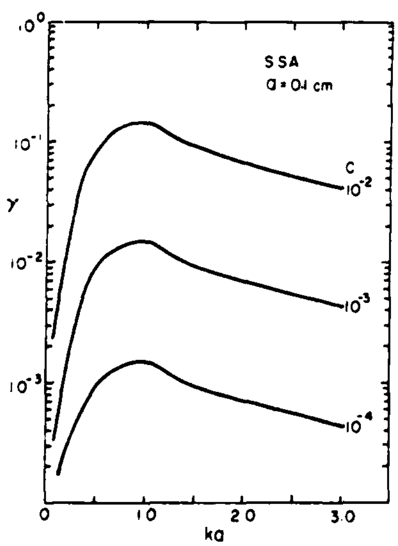

Fig. 4. The coherent attenuation constant γ versus ka for $\varepsilon_r = \varepsilon_r(\lambda)$ using SSA.

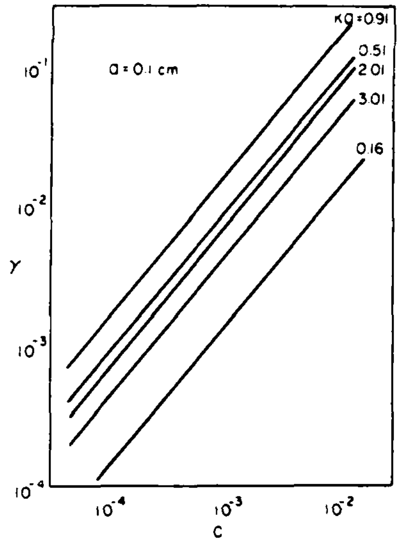

Fig. 5. The normalized attenuation constant γ versus concentration c for different values of ka using the WSA.

where Q_{ext} is the normalized (with respect to πa^2) extinction cross section of a sphere of radius a. *Brown* [1980] has shown recently that under the conditions of $c \to 0$ and no correlation between particles, the Foldy-Twersky integral equation [see *Ishimaru*, 1978] for the coherent field reduces to (19) above and thus does not account for the effects of multiple scattering. To demonstrate that (19) and the theoretical procedure given in this paper lead to identical results, we have compared computations for a monodisperse rain medium. The rain medium is assumed to consist of spherical water drops at concentrations in the range $10^{-4} < c < 10^{-2}$ which encompasses even the heaviest rainfall conditions. In Figure 3 the attenuation constant $\gamma = 4\pi K_2/K_1$ is shown as a function of freqency or ka using the WSA, which is to be compared to Figure 4 which uses (19) or the SSA. The refractive index of water is taken from *Ray* [1972] assuming a temperature of 5°C. Note that both solutions yield nearly identical results. In Figure 5 we show γ versus concentration (or equivalently, rainfall rate) at a number of fixed frequencies or ka values using the WSA. The straight line relationship reflects an attenuation versus concentration relationship of a power law form. *Olsen et al.* [1978] have used (19) to derive the power law relation between attenuation and rainfall rate (see their Figure 3).

At higher concentrations, pair correlation effects become important. In the Rayleigh or low-frequency limit, *Twersky* [1978*b*] has given an expression for

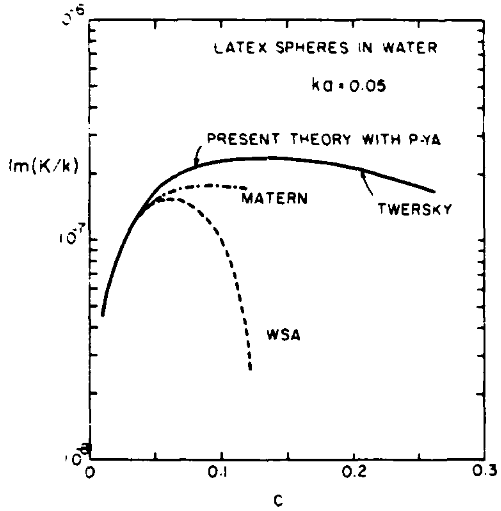

Fig. 6. The coherent attenuation Im (K/k) versus concentration c at $ka = 0.05$ for latex spheres in water.

Im (K/k) by considering the leading effects of the pair correlation:

$$\mathrm{Im}\,(K/k) = c(ka)^3[(\varepsilon_r - 1)/(\varepsilon_r + 2)]^2 W \tag{20}$$

where W is the packing factor given by

$$W = \frac{(1-c)^4}{(1+2c)^2} = 1 + 24c\int_0^{\infty} x^2[g(x) - 1]\,dx \tag{21}$$

The above integral can be intergrated analytically to give W when $g(x)$ is given by the P-YA. In Figure 6 we show Im (K/k) versus concentration for a fixed frequency or $ka = 0.05$. The random medium is assumed to consist of latex spheres ($\varepsilon_r = 2.26$) imbedded in water and corresponds to the experimental setup used by Ishimaru (A. Ishimaru, personal communication, 1981) for his optical propagation experiments. The agreement between (20) and the computations using the P-YA is excellent as expected while both the WSA and the Matern model fail (Im $K < 0$) for $c > 0.125$. However, the Matern model is superior to the WSA as expected. In Figure 7 we show similar results at $ka = 0.56$ for which Ishimaru (personal communication, 1981) has experimental results (latex sphere diameter = 0.107μ, $\lambda = 0.6\mu$) at concentration values of 0.01 and 0.1. Note that the SSA overestimates the attenuation constant even for c as low as 0.01; as c increases, the deviation from the P-YA is significant. For low values of c, the P-YA, WSA, and Matern model are all in excellent agreement. Even at $c = 0.10$, all three pair correlation models are in reasonably good agreement with the experimental value. For $c > 0.125$ the Matern model, as well as the WSA, deviates considerably from the P-YA. Even though experimental values for $c > 0.1$ are as yet unavailable, we feel the P-YA will continue to predict the correct behavior for values of c as high as 0.35. In Figure 8 the attenuation constant is plotted versus ka for a fixed concentration $c = 0.209$ using the SSA, WSA, and the P-YA. Values for the WSA for $ka < 0.75$ are not shown since the solution fails (Im $K < 0$) in this region. However, as ka increases, it appears that the WSA tends to merge with the P-YA. The SSA, on the other hand, consistently overestimates the attenuation over the full range of ka values.

4. CONCLUSIONS

This paper analyzes the effects of multiple scattering on the coherent wave as it propagates through a discrete random medium consisting of pair-correlated scatterers. At low concentrations ($c \leq 1\%$), multiple scattering effects are seen to be negligible, while at higher concentrations, suitable pair correlation models must be assumed. Computations are presented using the well-stirred approximation and the Percus-Yevick approximation as well as a model due to Matern. These computations are compared (and shown to be in excellent agreement)

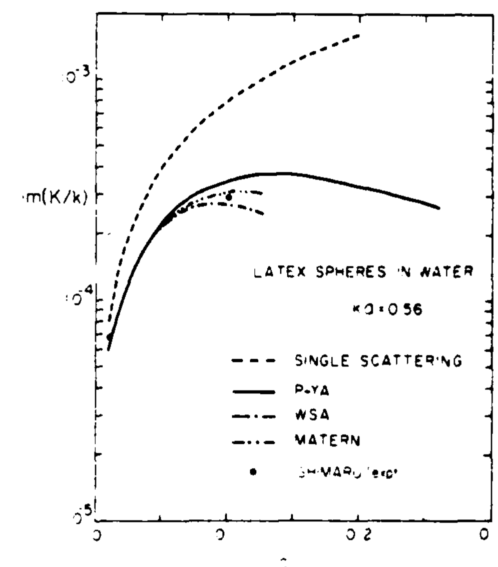

Fig. 7. The coherent attenuation Im (K/k) versus concentration c at $ka = 0.56$ for latex spheres in water using different models of pair correlation functions.

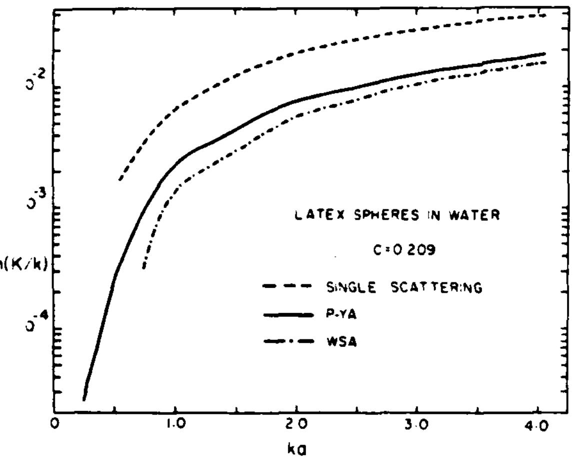

Fig. 8. The coherent attenuation Im (K, k) versus ka for $c = 0.209$ for latex spheres in water using SSA, P-YA, and WSA.

with recent experiments by Ishimaru (person . communication, 1981) at high scatterer concentrations ($c = 10\%$). At very low concentrations, $c \leq 1\%$, the WSA is compared (and shown to be in excellent agreement) with the so-called single scattering solution for a hypothetical monodisperse rain medium ($10^{-4} < c < 10^{-2}$). At these concentration levels the rain medium is sparse and the scatterers are essentially decorrelated so that multiple scattering effects, as far as the coherent wave attenuation is concerned, can be safely neglected.

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The Effects on Pair Correlation Function of Coherent Wave Attenuation in Discrete Random Media

V. N. BRINGI, MEMBER, IEEE, V. V. VARADAN, AND V. K. VARADAN

Abstract—**The Percus–Yevick approximation (P-YA) of pair correlation function for hard spheres is combined with the *T*-matrix formulation to study the coherent wave attenuation of electromagnetic wave propagation in a discrete random medium. The effect of the pair correlation function is seen to be significant at high fractional volumes of the discrete scatterers (≥ 0.125), but also depends on the frequency of the propagating wave—the effect being less at higher frequencies. The results are compared with previous calculations which employed the "well-stirred approximation" (WSA) for the pair correlation.**

INTRODUCTION

We consider the multiple scattering of electromagnetic waves by randomly distributed dielectric scatterers using the approach given by Varadan, Bringi, and Varadan [1]. This formulation uses the *T*-matrix [2] to characterize the scattering by a single isolated particle followed by configurational averaging techniques [3], [4]. Lax's quasi-crystalline approximation (QCA) [4] is used to truncate the resulting heirarchy of equations. Thus, only knowledge of the two particle (pair) correlation function is required. In [1], we assumed that the particles were hard (nonpenetrating) but otherwise uncorrelated. Talbot and Willis [5] refer to this as the "well-stirred approximation" (WSA). This yielded a set of "hole correction" integrals which were evaluated analytically and the extinction theorem was invoked to yield the dispersion relations characterizing the bulk or effective properties of the medium which was solved numerically [1], [6], [7]. Computations of the effective coherent wave attenuation as a function of frequency were presented in [1] for spherical and oblate spheroidal scatterers at concentrations (c) of 0.05, 0.1, and 0.2. The WSA leads to unphysical nulls in the plots of coherent attenuation as a function of frequency or concentration for $c \geqslant 0.125$ and in fact begins to fail for $c > 0.05$ in the low frequency or Rayleigh limit. These nulls, however, disappear at higher values of the nondimensional wave number ka (a being a characteristic dimension of the obstacle).

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V. N. Bringi is with the Department of Electrical Engineering, Colorado State University, Fort Collins, CO 80523.

V. V. Varadan is with the Wave Propagation Group, Boyd Laboratory, and the Department of Engineering Mechanics, The Ohio State University, Columbus, OH 43210.

V. K. Varaden is with the Wave Propagation Group, Boyd Laboratory, and the Department of Engineering Mechanics, The Ohio State University, Columbus, OH 43210.

In this communication, we wish to consider a more realistic model of the pair correlation function which is valid for higher values of concentration. The Percus–Yevick (P-YA) model [8] seems most suitable at the present time. Calculations employing P-YA here are compared with WSA results in [1]. The unphysical nulls in [1] disappear for lower values of ka, and at higher values of ka there is agreement with the results obtained in [1] and [7].

MULTIPLE SCATTERING FORMULATION

We consider $N(N \to \infty)$ rotationally symmetric oriented scatterers randomly distributed in a volume $V(V \to \infty)$ so that the number of particles per unit volume $n_0 = N/V$ is finite, see Fig. 1. Only the most important details that lead to the dispersion equation involving the pair correlation function are presented and for all intermediate steps, we refer the reader to [1].

Monochromatic, plane electromagnetic waves giving rise to an electric field $\vec{E}^0$ are assumed to propagate parallel to the rotational axis of symmetry of the scatterers (the z-axis), see Fig. 1. The field scattered by the ith scatterer is denoted by $\vec{E}_i^s$ so that the total field $\vec{E}$ at a point $\vec{r}$ outside the scatterers is given by

$$\vec{E}(\vec{r}) = \vec{E}^0(\vec{r}) + \sum_{i=1}^{N} \vec{E}_i^s(\vec{r}). \tag{1}$$

The field exciting the ith scatterer $\vec{E}_i^e$ is given by

$$\vec{E}_i^e(\vec{r}) = \vec{E}^0(\vec{r}) + \sum_{j \neq i}^{N} \vec{E}_j^s(\vec{r}); \quad a < |\vec{r} - \vec{r}_i| < 2a. \tag{2}$$

From (1) and (2), we note that

$$\vec{E}(\vec{r}) = \vec{E}_i^e(\vec{r}) + \vec{E}_i^s(\vec{r}) \tag{3}$$

so that the exciting and scattered fields must be defined in a self-consistent manner. These fields are expanded in a set of vector spherical functions as follows:

$$\vec{E}_i^e(\vec{r}) = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \sum_{\sigma=e}^{o} \{b_{\sigma m l}^{l} \,\mathrm{Re}\, \vec{M}_{\sigma m l}(\vec{r} - \vec{r}_i) + c_{\sigma m l}^{l} \,\mathrm{Re}\, \vec{N}_{\sigma m l}(\vec{r} - \vec{r}_i)\}; \quad a < |\vec{r} - \vec{r}_i| < 2a \tag{4}$$

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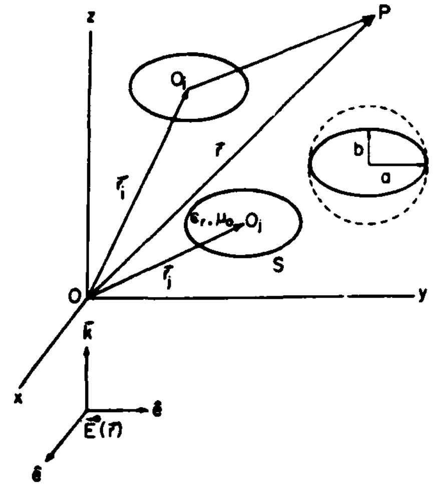

Fig. 1. Geometry of randomly distributed and aligned scatterers with dielectric constant ϵ_r excited by an incident plane wave $\vec E^0(\vec r)$. Each scatterer is assumed to be circumscribed by a spherical shell of radius a.

and

$$\vec E_l^s(\vec r) = \sum_l \sum_m \sum_\sigma \{B^l_{\sigma ml}\vec M_{\sigma ml}(\vec r - \vec r_l) + C^l_{\sigma ml}\vec N_{\sigma ml}(\vec r - \vec r_l)\}; \qquad |\vec r - \vec r_l| > 2a \tag{5}$$

where $\{b, c\}$ and $\{B, C\}$ are expansion coefficients of the exciting and scattered fields, respectively. The vector spherical functions $\{\vec M, \vec N\}$ are defined by Stratton [9].

These expansions are substituted into (2) with the following definition of the T-matrix of a single scatterer

$$\begin{bmatrix} B^j_{\sigma ml} \\ C^j_{\sigma ml} \end{bmatrix} = \begin{bmatrix} (T^{\sigma ml}_{\sigma' m' l'})^{11} & (T^{\sigma ml}_{\sigma' m' l'})^{12} \\ (T^{\sigma ml}_{\sigma' m' l'})^{21} & (T^{\sigma ml}_{\sigma' m' l'})^{22} \end{bmatrix} \cdot \begin{bmatrix} b^j_{\sigma' m' l'} \\ c^j_{\sigma' m' l'} \end{bmatrix} \tag{6}$$

(where the T-matrix is independent of the position of the scatterers) which results in an equation for the exciting field coefficients $\{b, c\}$ alone. This equation is averaged over the position of all scatterers where the QCA [4] is involved, and we arrive at an equation for the configurational average $\langle b^l\rangle_l$ and $\langle c^l\rangle_l$ of the exciting field coefficients with one particle held fixed.

We assume that this average field (the coherent field) propagates in a medium with an effective complex wavenumber $\vec K = (K_1 + iK_2)\hat k$ in the direction of the original incident field in the discrete medium. Thus, we obtain

$$\langle b^l_{\sigma ml}\rangle_l = i^l Y_{\sigma ml} e^{i\vec K \cdot \vec r_l} \tag{7}$$

and

$$\langle c^l_{\sigma ml}\rangle_l = i^l Z_{\sigma ml} e^{i\vec K \cdot \vec r_l} \tag{8}$$

where $\{Y, Z\}$ are expansion coefficients of the average exciting field. The dispersion equations then take the following form:

$$\begin{aligned} i^{n'} Y_{01n'} &= \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \sum_{\lambda=|n-n'|}^{n+n'} i^p(-i)^\lambda (JH)_\lambda \{Y_{01p}[(T^{01n}_{01p})^{11} \\ &\quad \cdot \psi_{00}(n, n', \lambda) + (T^{e1n}_{01p})^{21}\chi_{e0}(n, n', \lambda)] \\ &\quad + Z_{e1p}[(T^{01n}_{e1p})^{12}\psi_{00}(n, n', \lambda) \\ &\quad + (T^{e1n}_{e1p})^{22}\chi_{e0}(n, n', \lambda)]\} \end{aligned} \tag{9}$$

$$\begin{aligned} i^{n'} Z_{e1n'} &= \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \sum_{\lambda=|n-n'|}^{n+n'} i^p(-i)^\lambda (JH)_\lambda \{Y_{01p}[(T^{01n}_{01p})^{11} \\ &\quad \cdot \chi_{0e}(n, n', \lambda) + (T^{e1n}_{01p})^{21}\psi_{ee}(n, n', \lambda)] \\ &\quad + Z_{e1p}[(T^{01n}_{e1p})^{12}\chi_{0e}(n, n', \lambda) \\ &\quad + (T^{e1n}_{e1p})^{22}\psi_{ee}(n, n', \lambda)]\} \end{aligned} \tag{10}$$

where $(JH)_\lambda$ is an integral given by

$$\begin{aligned} (JH)_\lambda(K, k, c) &= \frac{6c}{(ka)^2 - (Ka)^2}\,[2ka\,j_\lambda(2Ka)h_\lambda'(2ka) \\ &\quad - 2Ka\,h_\lambda(2ka)j_\lambda'(2Ka)] \\ &\quad + 24c\int_{x=1}^{\infty} x^2[g(x) - 1]\,h_\lambda(kx)j_\lambda(Kx)\,dx \end{aligned} \tag{11}$$

and the functions $\{\psi, \chi\}$ are defined in [7].

In the above equation, $c = 4\pi a^3 n_0/3$ is the effective "spherical" concentration. In (9) and (10), ψ and χ are independent of k and K, and expressions for them may be found in [1] and [7]. In (11), j_λ and h_λ are the spherical Bessel and Hankel functions and $g(x)$ is the pair correlation function that is discussed below.

PERCUS–YEVICK APPROXIMATION FOR THE PAIR CORRELATION FUNCTION

It is well known that at high concentrations the effects of pair correlation become important. Following the notation in [1], the conditional probability distribution $p(\vec r_j | \vec r_l)$ is expressed as

$$p(\vec r_j | \vec r_l) = \begin{cases} \dfrac{1}{V}\,g(x); & x > 1 \\ 0; & x < 1 \end{cases} \tag{12}$$

where $x = |\vec r_l - \vec r_j|/2a$. The Percus–Yevick integral equation

[8] for hard spheres is given by

$$\tau(\vec{x}) = 1 + n_0 \int_{|\vec{x}'|<2a} \tau(\vec{x}')\, d\vec{x}' - n_0 \int_{\substack{|\vec{x}'|<2a \\ |\vec{x}-\vec{x}'|>2a}} \tau(\vec{x}')\tau(\vec{x}-\vec{x}')\, d\vec{x}' \tag{13}$$

The function $\tau(x)$ is related to the direct correlation function $C(\vec{x})$ of Ornstein and Zernike and the pair distribution function $g(\vec{x})$ [9]:

$$\begin{aligned} g(\vec{x}) &= \tau(\vec{x}); & |\vec{x}| &> 2a \\ g(\vec{x}) &= 0; & |\vec{x}| &< 2a \\ C(\vec{x}) &= \tau(\vec{x}); & |\vec{x}| &< 2a \\ C(\vec{x}) &= 0; & |\vec{x}| &> 2a. \end{aligned} \tag{14}$$

For hard spheres, the direct correlation function $C(\vec{x})$ is known explicitly and Wertheim [10] has obtained a series solution for $g(x)$ as a function of concentration when $g(x)$ is radially symmetric. Throop and Bearman [11] have used the Wertheim result and provided tabulated values of $g(x)$ as a function of x for several values of c.

The highest concentration for which the tabulated results can be used is $c = 0.26$. Beyond this concentration $g(x)$ oscillates significantly from its asymptotic value of 1 for $x > 4$. Thus, the integral in (11) cannot be evaluated accurately.

At low values c, $g(x) \simeq 1$ and hence the integral in (11) is negligible and the remaining term is simply the WSA which was used in [1]. Thus, (11) can be regarded as a modified "hole correction integral" and is of the same form used by Twersky [12], [13]. Talbot and Willis [5] have also suggested models of the pair correlation function given by Matern which are valid for $c < 0.125$. The advantage of the Matern model is that it is completely analytic.

NUMERICAL COMPUTATIONS

In the Rayleigh limit, Twersky [13] has given an expression for K_2/k by considering the leading effects of the pair correlation function:

$$K_2/k = \mathrm{Im}\,(K/k) = c(ka)^3 \left\{\frac{\epsilon_{r-1}}{\epsilon_{r+2}}\right\}^2 W \tag{15}$$

where ϵ_r is the dielectric constant of the scatterer and W is the packing factor given by

$$W = \frac{(1-c)^4}{(1+2c)^2} = 1 + 24c \int_0^\infty x^2[g(x) - 1]\, dx. \tag{16}$$

In Fig. 2, we compare Twersky's result for the coherent attenuation with computations for $ka = 0.05$ and $\epsilon_r = 3.168$ as a function of concentration. Also shown is the result assuming the WSA. It is clear that the WSA fails for $c \geqslant 0.05$ while the present calculations is in good agreement with (15).

In Figs. 3, 4, and 5, calculations using the P-YA model are presented as a function of ka for $c = 0.2$ for spheres and

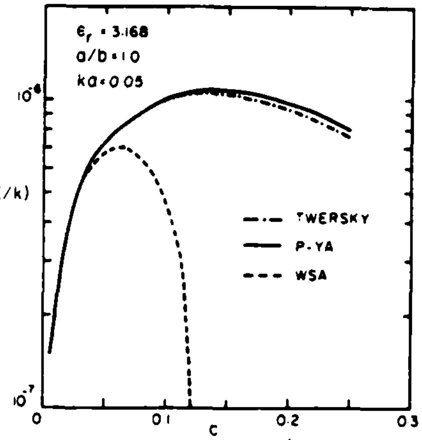

Fig. 2. Coherent wave attenuation, Im (K/k) as a function of concentration for dielectric spheres.

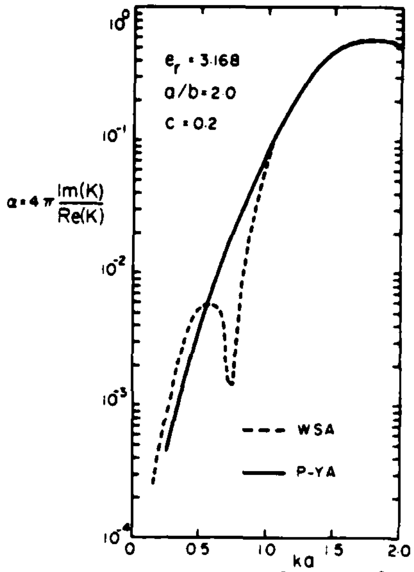

Fig. 3. Attenuation coefficient α as a function of ka for spherical dielectric scatterers.

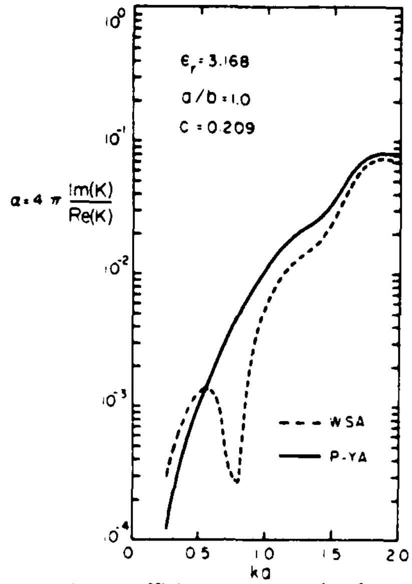

Fig. 4. Attenuation coefficient α versus ka for oblate spheroidal scatterers.

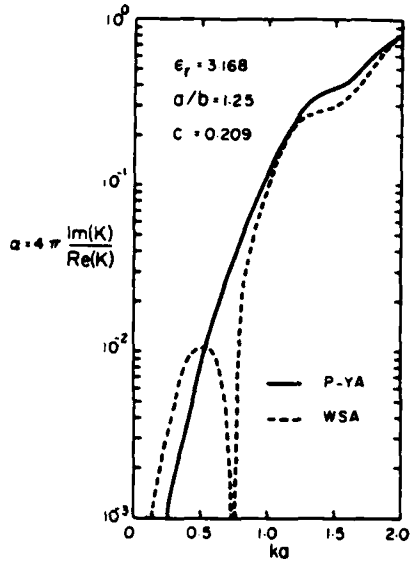

Fig. 5. Attenuation coefficient α versus ka for oblate spheroidal scatterers.

oblate spheroids. Equations (9) and (10) were solved numerically to find $K(k)$ using analytical results at low frequencies (Rayleigh limit) as initial guesses in a root searching algorithm. For purposes of comparison, [1, Figs. 3, 5, and 6] using just the WSA, are reproduced. It is clear that the nulls appearing as a result of the WSA are wrong and unphysical. However, it is interesting to observe that for $ka > 1.5$, it appears that the results obtained using the WSA alone tend to the P-YA result. Note that in [1, Fig. 3] the curves marked $c = 0.05$ and $c = 0.1$ must be interchanged.

CONCLUSION

Calculations using the Percus-Yevick (P-YA) pair correlation function for hard spheres are compared with previous computations [1] using the well stirred approximation. The effects of pair correlation are seen to be significant for high values of scatterer concentrations $c > 0.05$, although this effect seems to decrease with increasing frequency. It is clear that the WSA cannot be used for arbitrary concentrations and some of the results presented in [1], [6], and [14]-[16] are thus unphysical.

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Multiple scattering theory for waves in discrete random media and comparison with experiments

V. K. Varadan,[1] *V. N. Bringi*,[2] *V. V. Varadan*,[1] *and A. Ishimaru*[3]

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Attenuation of electromagnetic waves by a random distribution of pair-correlated dielectric spheres is studied as a function of frequency and volume concentration of spheres. The main aim of this paper is to compare theoretical results obtained using a self-consistent multiple-scattering formulation and measured values of attenuation for latex spheres in water. The agreement between theory and experiment is very good.

INTRODUCTION

A discrete random medium is defined here as a random distribution of a large number of identical pair-correlated scatterers embedded in an infinite homogeneous matrix medium. The fractional volume concentration c is assumed uniform, and the effects of pair correlation are described by the radial distribution functions arising in the statistical-mechanical treatment of dense gases and liquids. We consider plane wave propagation through such a model random medium with attention being focused on the coherent (or ensemble averaged) wave attenuation. The discrete random medium is then characterized by an effective, complex wave number K, or equivalently, by an effective dielectric constant. If the scatterers and the matrix medium are assumed lossless, then the coherent wave attenuation is caused by scattering, and it is of great interest to determine the wave frequency and scatterer concentration for which multiple-scattering effects dominate. Extensive work by *Twersky* [1977, 1978*a*, *b*, *c*, 1982] has laid the foundation for multiple-scattering theory in discrete random media. A related approach using the T matrix of a single scatterer [*Varadan and Varadan*, 1980*a*] together with configurational averaging procedures has been used by the authors to develop a computational method for handling acoustic, electromagnetic, and elastic waves [*Varadan et al.*, 1979, 1982; *Varadan and Varadan*, 1980*a*, *b*; *Bringi et al.*, 1981, 1982*a*, *b*]. *Lax*'s [1952] quasi-crystalline approximation (QCA) is used in conjunction with various models for the radial distribution function enabling passage to an average dielectric medium whose properties depend on frequency, scatterer concentration, and the dielectric constants of both the scatterers and the matrix medium. The intent of this paper is to compare theoretical calculations of coherent attenuation with optical experiments conducted recently by *Ishimaru and Kuga* [1982]. Preliminary comparisons between theory and experiment have been reported [*Bringi et al.*, 1982*a*, *b*], where the single-scattering approximation is compared with multiple-scattering calculations using various forms for the radial distribution function (e.g., Matern and Percus-Yevick models).

In this paper we provide an outline of the multiple-scattering theory followed by a brief discussion of the self-consistent radial distribution function used in the computations. This is followed by calculation of coherent wave attenuation as a function of scatterer concentration ($0 < c < 0.4$) for a number of wave frequencies ($0 < ka < 7$) coinciding with the experimental setup of *Ishimaru and Kuga* [1982]. Finally, previous computations presented by *Bringi et al.* [1981] corresponding to the experiments of *Hawley et al.* [1967] are discussed in the light of the results presented here.

OUTLINE OF THEORY

Consider an incident TEM wave propagating in an infinite lossless, background medium of ε_m, μ_0 which

[1] Wave Propagation Group, Boyd Laboratory, Department of Engineering Mechanics, and Atmospheric Sciences Program, Ohio State University, Columbus, Ohio 43210.

[2] Department of Electrical Engineering, Colorado State University, Ft. Collins, Colorado 80523.

[3] Department of Electrical Engineering, University of Washington, Seattle, Washington 98195.

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is filled with a random distribution of identical dielectric spheres of ε, μ_0. The total dielectric field at any point in the background medium is the sum of the incident field and the fields scattered by all the scatterers,

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{\text{inc}}(\mathbf{r}) + \sum_{i=1}^{N} \mathbf{E}_i^s(\mathbf{r} - \mathbf{r}_i) \qquad (1)$$

where $\mathbf{E}_i^s(\mathbf{r} - \mathbf{r}_i)$ is the field scattered by the ith scatterer at the observation point $\mathbf{r}$ and $\mathbf{r}_i$ is the position of the ith scatterer. The field that excites the ith scatterer, however, is the incident field plus the fields scattered from all the other scatterers. The term exciting field, $\mathbf{E}^e$, is used to distinguish between the field actually incident on a scatterer and the external incident field $\mathbf{E}^{\text{inc}}$ produced by a source at infinity. Thus at a point $\mathbf{r}$ in the vicinity of the ith scatterer, the exciting field is expressed as

$$\mathbf{E}_i^e(\mathbf{r}) = \mathbf{E}^{\text{inc}}(\mathbf{r}) + \sum_{j \neq 1}^{N} \mathbf{E}_j^s(\mathbf{r} - \mathbf{r}_j) \qquad (2)$$

From (1) and (2) we note that

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_i^e(\mathbf{r}) + \mathbf{E}_i^s(\mathbf{r}) \qquad (3)$$

so that the exciting and scattered fields must be defined in a self-consistent manner. Waterman's T matrix formalism is used to relate the exciting and scattered field expansion coefficients (the fields are expanded using the vector spherical functions as a basis). Configurational averaging is performed in (2) as described by *Varadan et al.* [1979]. The T matrix together with the QCA is used to generate a homogeneous system of equations whose singular solutions yield the average propagation constant ($K = K_1 + jK_2$) for the discrete random medium. The system of equations is given below, and the reader is referred to *Bringi et al.* [1981], *Varadan et al.* [1979, 1982], and *Varadan and Varadan* [1980*a*] for full details.

$$Y_m = \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \sum_{\lambda=|n-m|}^{\infty} (JH)_\lambda \cdot [Y_p T_{np}^{11} \psi(n, m, \lambda) + Z_p T_{np}^{22} \psi(n, m, \lambda)] \qquad (4a)$$

$$Z_m = \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \sum_{\lambda=|n-m|}^{\infty} (JH)_\lambda \cdot [-Y_p T_{np}^{11} \chi(n, m, \lambda) + Z_p T_{np}^{22} \psi(n, m, \lambda)] \qquad (4b)$$

where

$$(JH)_\lambda = \frac{6c}{(ka)^2 - (Ka)^2} \cdot [2ka j_\lambda(2Ka) h_\lambda'(2ka) - 2Ka h_\lambda(2ka) j_\lambda'(2Ka)] + 24c \int_1^\infty x^2 \{g(x) - 1\} h_\lambda(2kax) j_\lambda(2Kax)\, dx \qquad (4c)$$

In (4*a*) and (4*b*), Y_m and Z_m are the unknown amplitudes of the average exciting field, T_{np} refers to the elements of the T matrix for a sphere, and the functions ψ and χ are defined by *Bringi et al.* [1981]. In (4*c*), $g(x)$ is the radial distribution function; j_λ and h_λ are the spherical Bessel functions, and the primes denote differentiation with respect to the argument; k is the wave number of the background medium; a is the scatterer radius; and c is the fractional concentration of the scatterers. Assuming that $g(x)$, ka, c, and the T matrix are known, the singular value of the coefficient matrix generated from (4) can be solved for the average propagation constant $K = K_1 + jK_2$.

RADIAL DISTRIBUTION FUNCTION

The discrete random medium is considered as a statistical ensemble of impenetrable spheres. In the statistical mechanics literature this is synonymous with an ensemble of 'hard' spheres. The radial distribution $g(r)$ is defined in terms of the two-particle joint probability density $p(\mathbf{r}_j | \mathbf{r}_i)$ defined as

$$\begin{aligned} p(\mathbf{r}_j | \mathbf{r}_i) &= \frac{1}{V} g(|\mathbf{r}_j - \mathbf{r}_i|) \qquad |\mathbf{r}_j - \mathbf{r}_i| \geq 2a \\ p(\mathbf{r}_j | \mathbf{r}_i) &= 0 \qquad |\mathbf{r}_j - \mathbf{r}_i| < 2a \end{aligned} \qquad (5)$$

Equation (5) implies that the particles are hard (no interpenetration) and the excluded volume is a sphere of radius a although the particles themselves may be nonspherical. The function $g(r)$, ($\mathbf{r} = \mathbf{r}_{ij}$), is called the pair correlation function and depends only on $|\mathbf{r}_{ij}| = |\mathbf{r}_j - \mathbf{r}_i|$ because of translational invariance of the system under consideration. Several theories and calculations are available for determining $g(r)$, namely, the hypernetted-chain equation, the Percus-Yevick approximation (P-YA), the self-consistent approximation, Monte Carlo calculations, etc. These approaches are described by *McQuarrie* [1976], but the reader is referred to any standard text on statistical mechanics for full details.

The pair correlation function for an ensemble of particles depends on the nature and range of the interparticle forces. The average of several measurements of a statistical variable that characterizes an ensemble will depend on the pair correlation function. To obtain expressions for the pair correlation function, one needs a description of the interparticle forces. In our case, we assume that the scatterers behave like effective hard (impenetrable) spheres. *Percus and Yevick* [1958] have obtained an approximate integral equation for the pair correlation function of a classical fluid in equilibrium. *Wertheim* [1963] has obtained a series solution of the integral equation for an ensemble of hard spheres. The statistics of the fluid are then the same as those of the ensemble of discrete hard particles that we are considering.

Although integral expressions for the correlation functions also result in a hierarchy, *Percus and Yevick* [1958] have truncated the hierarchy by making certain approximations that result in a self-consistent relation between the pair correlation function $g(r)$ and the direct correlation function $C(r)$. The direct correlation can be defined to be the direct effect of scatterer 1 on 2 which roughly has the range of the interparticle potential and also an indirect effect due to the effect of 1 on 3 which in turn has an effect on 2. Since 3 can be any scatterer of the system, it will include a sum on all scatterers and an integration over the volume of the system. *Fisher* [1965] comments that the P-YA is a strong statement of the extremely short range nature of the direct correlation function. For impenetrable spheres, the range of the interparticle potential is $2a$. The integral equation has the form [see *Percus and Yevick*, 1958] given by

$$\tau(r) = 1 + n_0 \int_{r < 2a} \tau(r')\, dr' - n_0 \int_{\substack{r' < 2a \\ |r - r'| > 2a}} \tau(r')\tau(r - r')\, dr' \tag{6}$$

where

$$\begin{aligned} \tau(r) &= g(r) & r &> 2a \\ g(r) &= 0 & r &< 2a \\ \tau(r) &= -C(r) & r &< 2a \\ C(r) &= 0 & r &> 2a \end{aligned} \tag{7}$$

Wertheim [1963] has solved the integral equation by Laplace transformation that results in an analytic expression for $C(r)$ in the form

$$C(r) = -(1 - \eta)^{-4}[(1 + 2\eta)^2 - 6\eta(1 + \tfrac{1}{2}\eta)^2 r + \eta(1 + 2\eta)^2 r^3\ 2] \qquad \eta = c\ 8 \tag{8}$$

where c is the effective spherical concentration of the particles. The P-YA fails as the concentration approaches the close packing factor for spheres and is expected to be good for $c < 0.3$ or 0.4.

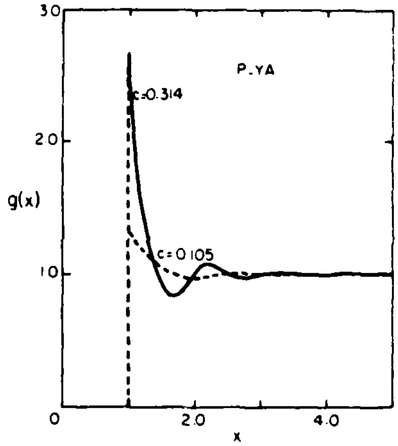


Fig. 1. The Percus-Yevick (P-YA) pair correlation function for 'hard' spheres.

Equation (8) can be substituted back into (6) to yield a series solution for $g(r)$ in the form [see *Wertheim*, 1963]

$$g(r) = \sum_{n=1}^{\infty} g_n(r) \tag{9}$$

where

$$g_n(r) = \frac{1}{24 r_i} \int e^{t(r-n)}[L(t) | S(t)]^n t\, dt \tag{10}$$

$$S(t) = (1 - \eta^2)t^3 + 6\eta(1 - \eta)t^2 + 18\eta^2 t - 12\eta(1 + 2\eta) \tag{11}$$

$$L(t) = 12\eta[(1 + \eta^2)t + (1 + 2\eta)] \tag{12}$$

Throop and Bearman [1965] have tabulated $g(r)$ as a function of r for values of $\eta = c/8$. A few representative plots of the Percus-Yevick (P-Y) pair correlation function are shown in Figure 1.

Another approximation to $g(r)$ is the hypernetted-chain equation (HNC) which differs from the P-YA in that the direct correlation function $C(r)$ has a longer range. In general, P-YA is expected to be better than HNC. The P-Y and HNC results are good approximations to $g(r)$ at low concentrations but are appreciably in error at high concentrations.

At low concentrations, a series expansion can be obtained from (6) by iterating in powers of n_0. This power series in density is the virial expansion and has been used at low frequencies by *Twersky* [1977, 1978*a*, *b*, *c*], *Bringi et al.* [1981, 1982*a*, *b*], and *Varadan et al.* [1982]. For the leading terms, (6) is

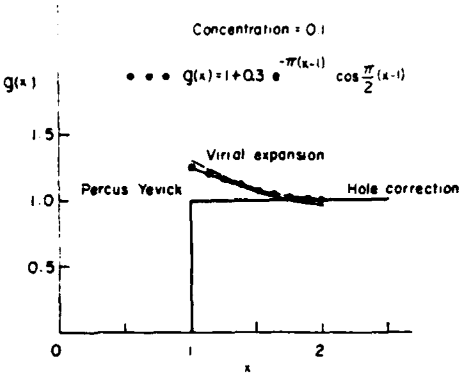


Fig. 2. The P-YA, hole correction, virial expansion pair correlation functions, and pair correlation functions based on module histogram for impenetrable spheres.

written as [see *Percus and Yevick*, 1958]

$$\tau_{12} = 1 + n_0 \int f_{23}\tau_{23}(\tau_{13} - 1)\, d\mathbf{r}_3 + n_0 \int f_{23} f_{13} \tau_{23}\tau_{13}\, d\mathbf{r}_3 \tag{13}$$

with

$$f_{ij} = -1 \qquad |\mathbf{r}_{ij}| < 2a$$
$$f_{ij} = 0 \qquad |\mathbf{r}_{ij}| > 2a$$

Commencing at $\tau_{12} = 1$, one can readily iterate a power series to obtain

$$\tau_{12} = 1 + n_0 \int f_{23} f_{13}\, d\mathbf{r}_3 + n_0^2 \iint [f_{23} f_{34} f_{14} + 2 f_{23} f_{34} f_{14} f_{24}]\, d\mathbf{r}_3\, d\mathbf{r}_4 + \cdots \tag{14}$$

For hard spheres, $g(r)$ can be determined exactly to $O(n_0^3)$ [see *Percus and Yevick*, 1958; *Twersky*, 1977, 1978*a*, *b*, *c*] given by

$$g(r) = 1 \qquad r > 2a$$
$$g(r) = 1 + \frac{4\pi}{3} a^3 n_0 \left(1 - \frac{3r}{4a} + \frac{r^3}{16a^3}\right) \qquad a < r < 2a \tag{15}$$
$$g(r) = 0 \qquad r < a$$

but a slight discrepancy has been observed in $O(n_0^4)$ [see *McQuarrie*, 1976] because of the approximation in the P-YA. The variation of $g(r)$ in terms of virial expansion is shown in Figure 2. At low frequencies, a relatively simpler form of $g(r)$ given in terms of trigonometrical functions as shown in Figure 2 may also be used.

At higher concentrations, the values of $g(r)$ using the P-YA are in appreciable error. The reason for this error is that when an approximate theory such as P-YA for $g(r)$ is used to calculate the pressure, two different values are obtained from the pressure equation and the compressibility equation:

$$p = \frac{1}{K_B T}\left\{n_0 - \int \left(e^{-V(\mathbf{r})/K_B T} - 1\right)\left[\int_0^{n_0} n' \tau(\mathbf{r}, n')\, dn'\right] d\mathbf{r}\right.$$
$$p = n_0 K_B T\left[1 + \int \frac{n_0}{6} \tau(\mathbf{r})\mathbf{r} \cdot \nabla e^{-V(\mathbf{r})/K_B T}\, d\mathbf{r}\right] \tag{16}$$

where K_B is the Boltzmann constant and T is the system temperature. For hard spheres,

$$V(\mathbf{r}) = \infty \qquad |\mathbf{r}| < 2a$$
$$V(\mathbf{r}) = 0 \qquad |\mathbf{r}| > 2a \tag{17}$$

which shows that the system is independent of temperature but depends only on number density or concentration. An exact theory for $g(r)$ should give the same calculated pressure from both equations. *Rowlinson* [1965] has suggested a method known as the self-consistent approximation (SCA) for optimizing the P-YA and removing the inconsistency in the two pressure equations (16) by assuming that the direct correlation function may be written as

$$C_{SCA}(r) = C_{P\text{-}YA}(r) + \Phi C_{HNC}(r) \tag{18}$$

where Φ is an adjustable parameter which depends on concentration but not on separation r and C_{SCA} is the self-consistent approximation of the direct correlation function. Using (7) and (18), the two pressure equations (16) that depend on $g(r)$ are solved self-consistently by adjusting the function Φ and lead to an integral equation for $g(r)$ which is then solved numerically. At higher concentrations, the SCA is thus an improvement over the P-YA and HNC approximation. From (18) it is obvious that when $\Phi = 0$, (18) reduces to the P-YA, whereas if $\Phi = 1$, it gives the HNC approximation. *Reed and Gubbins* [1973] have provided tabulated values of $g_{SCA}(r)$ for $0.0524 \le c \le 0.417$ and for $1.0 \le r \le 5.0$, and a general computer program for computing g_{SCA} is given by *McQuarrie* [1976]. Tabulated values of $g_{SCA}(r)$ are also given by *Watts and Henderson* [1969], but they are not useful for computations because the values of g are given only for a short range of r. Monte Carlo calculations for $g(r)$ are done, and their values are tabulated by *Barker and Henderson* [1971]; they cannot be used for computations for the same reasons. However, the values given by *Barker and Henderson* [1971] provide a very good check for higher-

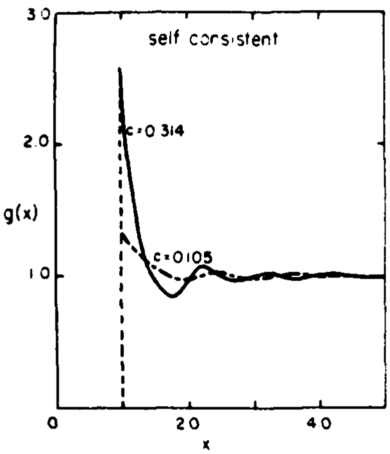


Fig. 3. The self-consistent pair correlation function for 'hard' spheres.

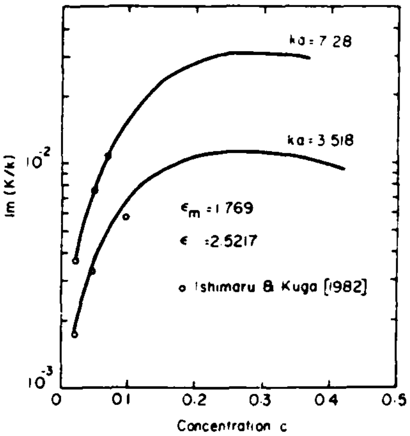


Fig. 5. Im (K/k) versus concentration using the self-consistent radial distribution function.

order virial coefficients. The computer program given in the work by *McQuarrie* [1976] is thus preferable for computational purposes. A representative plot of $g_{SCA}(r)$ versus r is shown in Figure 3.

COMPARISON WITH EXPERIMENTS

Extensive propagation experiments have been conducted by *Ishimaru and Kuga* [1982] where the discrete random medium comprises a collection of nearly identical latex spheres embedded in water and illuminated with a HeNe laser ($\lambda = 0.6328$ m). The coherent wave attenuation is measured over a range of concentrations (up to 40%) and for a number of equivalent particle sizes (low, Mie, and optical ka values). Computations of coherent attenuation K/k as a function of frequency (or ka) and concentration are performed by determining the singular values of the coefficient matrix obtained from (4a) and (4b) using the self-consistent radial distribution function. $(K/k)^2$ is the effective dielectric constant of the bulk random medium relative to the background water medium. Figure 4 shows computations of Im (K/k) as a function of concentration for ka values of 0.529 and 0.681 together with the experimental points of *Ishimaru and Kuga* [1982]. Note that the original experimental points have been converted to Im (K/k) by using a normalization value described by *Ishimaru and Kuga* [1982]. The comparison between theory and experiment is seen to be excellent even at high concentrations. In Figure 5 we show computations of Im (K/k) versus concentration of ka values of 3.518 and 7.28. Again the experimental points are seen to be in excellent agreement with the computations. Even though the experiments were not conducted at $c > 10\%$ at these ka values, the behavior of Im (K/k) is correctly reproduced by the self-consistent radial distribution function. In Figure 6 we compare our computations with the experimental results of *Ishimaru and Kuga* [1982]. The parameter $\gamma = 2K_2/n_0\sigma_t$ where $K_2 = \mathrm{Im}\,(K)$, n_0 is the number density, and σ_t is the extinction cross section of a single sphere. We remark that $n_0\sigma_t$ is the approximate value of the attenuation at very low concentrations. Hence the parameter γ is the attenuation at any concentration normalized to the low concentration limit. Thus, as $c \to 0$, $\gamma \to 1$. Further, as explained by *Ishimaru and Kuga* [1982], since the latex spheres used in the ex-

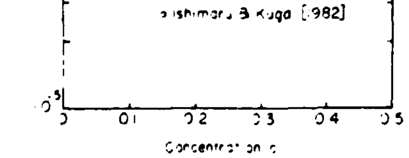
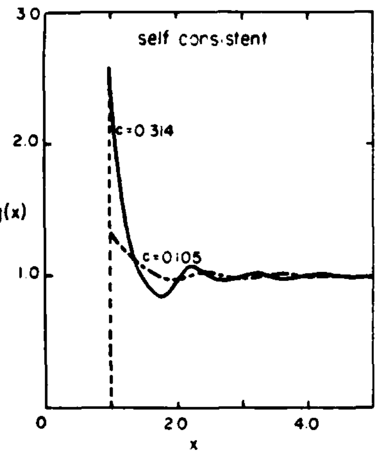


Fig. 4. Im (K/k) versus concentration using the self-consistent radial distribution function.



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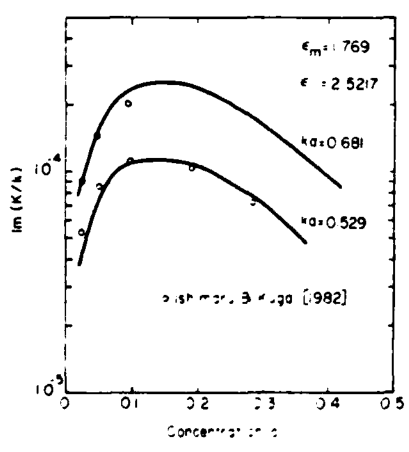

Fig. 4. Im (K/k) versus concentration using the self-consistent radial distribution function.

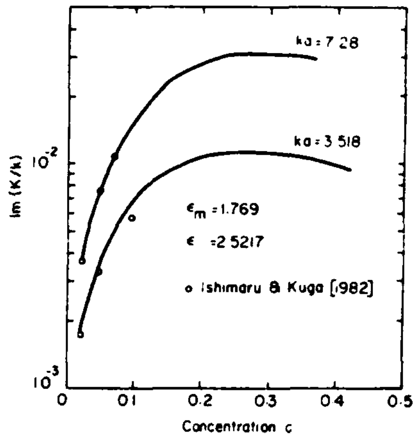

Fig. 5. Im (K/k) versus concentration using the self-consistent radial distribution function.

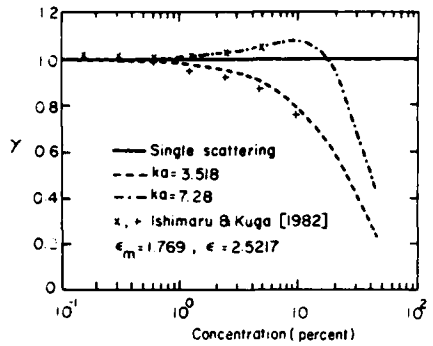

Fig. 6. Plot of $\gamma = 2$ Im $K/n_0\sigma_t$ versus concentration where n_0 is the number density and σ_t is the extinction cross section of a single sphere with relevant ka.

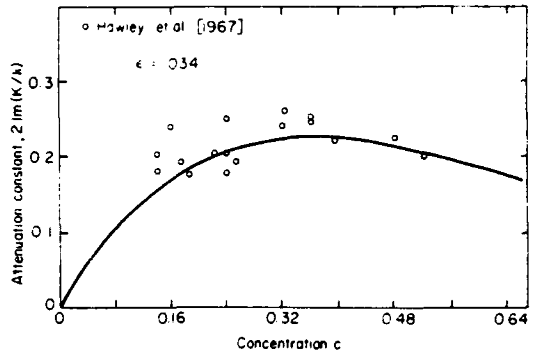

Fig. 7. Attenuation constant 2 Im (K/k) versus concentration for $ka = 11.8$ using the 'well-stirred' (effective concentration) radial distribution function. The experimental points are taken from *Hawley et al.* [1967].

periment were not exactly identical, the measured value of the attenuation at $c \leq 0.0038$ did not exactly correspond to $n_0\sigma_t$ where σ_t was computed at the average measured radius of the sphere. In order to take care of this discrepancy, the measured value of the attenuation α at low values of c was used to define the effective radius a_{eff} of a sphere which predicts the correct attenuation in the single-scattering approximation. All experimental results were normalized with respect to $\alpha_0 = n_0\sigma_t(a_{eff})$ and $\gamma = \alpha/\alpha_0$. Thus in all comparisons we have also normalized the computed value of α by α_0 using the effective radius supplied in Table I of *Ishimaru and Kuga* [1982]. The computations are in excellent agreement with experiment for ka values of 3.518 and 7.28 (see Figure 6). Note the sharp decrease in γ for $c \geq 10\%$, especially for the ka value of 7.28. The dramatic importance of multiple scattering is seen clearly from Figure 6 for $c > 1\%$: this value of c, however, depends on ka.

In our previous paper [*Bringi et al.*, 1982a, b] we observed that at higher values of ka, a less rigorous form for the radial distribution function $g(r) = 1/(1-c)$ may suffice even at relatively high concentrations ($c > 40\%$). It is important to note that at wavelengths comparable to obstacle size and higher, the scattering is mostly in the forward direction. Thus, in this case, repeated scattering should not be important, since the backscattered wave is significantly smaller than the forward scattered wave. This may explain why the computations shown in Figure 7 for $ka = 11.8$ are in reasonable agreement with the experiment done by Hawley et al. at higher values of concentrations. It should be noted that the P-YA and SCA fail as the concentration approaches the close packing factor for spheres and are expected to be good for $c < 0.42\%$. Thus, based on the results presented in Figure 7, the coherent wave attenuation is less sensitive to the precise form of the pair correlation function at high concentrations for relatively high values of ka.

CONCLUSIONS

This paper has highlighted the comparison between theory and experiments for coherent wave attenuation in a discrete random medium. Multiple-scattering effects must be considered for volume concentrations exceeding around 1% especially for ka values of ≤ 10. At higher concentrations ($c \leq 40\%$) and for $ka \leq 10$ the form of the radial distribution function is very important. At values of $ka \geq 10$, calculations show that the coherent wave attenuation is less sensitive to the precise form of the radial distribution function. Computations and experimental results suggest that at high ka values the simple 'well-stirred' approximation for the radial distribution function ($g(r) = 1/(1-c)$, $r < 2a$; $g(r) = 0$, $r > 2a$) may suffice even up to high concentrations. The theoretical procedure described in this paper agrees very well with experiments over a wide range of concentrations and ka values, $0 < c \leq 40\%$ and $0 < ka \leq 10$. Of course, these conclusions also depend on the scatterer properties relative to the embedding medium.

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Coherent attenuation of acoustic waves by pair-correlated random distribution of scatterers with uniform and Gaussian size distributions

V. K. Varadan, V. N. Bringi,[a] V. V. Varadan, and Y. Ma
Wave Propagation Group, Department of Engineering Mechanics, The Ohio State University, Columbus, Ohio 43210

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Acoustic wave attenuation due to multiple scattering in a two-phase medium consisting of a fluid with embedded rigid, fluid, or elastic particles of varying sizes is discussed. The formulation, involving the exciting and scattered fields of an incident acoustic plane wave, is based on the T-matrix method. The propagation features of coherent waves in the mixture are described by the dispersion equation which is derived by applying standard statistical approximations to the discrete random medium. Special attention is focused on the pair-correlation function between the scatterers using the self-consistent approximation (SCA) which seems better than the Percus-Yevick approximation (PYA) when the volume fraction becomes significant. Besides deriving low-frequency analytical results for coherent wave speed and attenuation, the dispersion equation has been solved numerically for higher frequencies for particles with uniform and Gaussian size distributions.

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INTRODUCTION

A study of wave propagation in a multi-component fluid medium is helpful in analyzing the gross properties of the mixture as a whole. The propagation and attenuation results derived from a typical investigation yield data that are valuable in various practical situations.

In formulating the theory for sound propagation in a fluid–particle mixture, two fundamentally different approaches have been followed in the literature. In the first approach, the solid particles are treated as scattering centers and a set of coupled equations is formulated to describe the multiple scattering. By computing the acoustic fields resulting from this scattering phenomenon, one arrives at formulas for the bulk parameters characterizing the medium. In the second approach, the field variables characterizing the medium are related through usual conservation laws and the response of the medium to incident acoustic pulses are derived from nonequilibrium thermodynamic considerations coupled with the hyperbolic nature of the governing system. The results accrued from both these approaches are largely seen to complement each other although a particular problem may entail selection of either of the procedures in preference to the other.

For suspended particles, multiple scattering theory combined with suitable statistical approximations lead to results that are valid for a wide range of frequencies. This approach takes account of the microscopic features of the scatterers, the effects of which are finally reflected in the coherent wave analysis. Although rigorous theoretical investigations of the acoustic wave attenuation in a fluid–particle medium can be traced to the work of Sewel,[1] systematic multiple scattering formulations capable of uniformly handling both the asymptotic and intermediate frequency ranges and based on realistic particle distributions and correlation effects were developed only in the last few years.[2–13]

In this paper the multiple scattering of acoustic waves by suspended particles is described using the T-matrix (Waterman[14]) to characterize the single scatterer response and a configurational average over the random positions of the particles. The method presented leads to a computational scheme that is suitable for scatterers of arbitrary shape, orientation, dense concentration, and at wavelengths comparable to scatterer size. The complex, effective wavenumber in the random medium is computed as a function of frequency. It is observed that the results crucially depend on the volume fraction c, and for $c > 0.1$ the effects of the pair correlation function for the given distribution is significant especially at lower frequencies. Accordingly, improved pair correlation functions using the self-consistent approximation (SCA) have been incorporated into the numerical algorithm. The SCA is a linear combination of the Percus–Yevick (PYA) and the Hypernetted Chain (HNC) approximation to the pair correlation function. This enables us to get numerical results for $c \sim 0.35$. Closed form expressions for the phase velocity and attenuation are presented in the long wavelength limit.

I. MULTIPLE SCATTERING FORMULATION

In the present study, the scatterers are assumed to be either rigid, fluid, or elastic whose properties differ from the embedding fluid medium, which, for all practical purposes, will be assumed here to be inviscid liquid akin to seawater. The present formulation deals with the multiple scattering effects of the solid phase on the coherent wave attenuation.

[a] V. N. Bringi is with the Department of Electrical Engineering, Colorado State University, Fort Collins, CO 80523.

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Consider N identical, arbitrarily shaped random scatterers with smooth boundary surface S in an infinite fluid medium which are referred to a coordinate system centered at O. The points O_i and O_j denote the centers of the ith and jth particles, respectively; and they are referred to the origin by the spherical polar coordinates (r_i, θ_i, ϕ_i). P is any point in the medium outside the scatterers. Let a time harmonic compressional wave of unit amplitude and frequency ω be incident on the scattering medium. Suppressing the time dependence of all quantities, we represent the pressure field corresponding to the incident wave in the form

$$\phi^0 = e^{ikz}, \tag{1}$$

where k is the acoustic wavenumber, $k = \omega/c_f$. The sound speed c_f is given by $(\lambda_f/\rho_f)^{1/2}$, where λ_f and ρ_f are the compressibility and density of the fluid, respectively. We denote the compressional and transverse wave speeds in the elastic scatterer by c_p and c_s given by $c_p = [(\lambda + 2\mu)/\rho]^{1/2}$ and $c_s = [(\mu/\rho)]^{1/2}$, respectively, where λ and μ are Lamé's constants and ρ is the density. For a fluid scatterer, $c_p = [(\lambda/\rho)]^{1/2}$, where λ and ρ are the compressibility and density of the fluid, respectively.

The total pressure field at any point outside the scatterers is given by,

$$\phi(\mathbf{r}) = \phi^0(\mathbf{r}) + \sum_{j=1} \phi_j^s(\mathbf{r} - \mathbf{r}_j), \tag{2}$$

where $\phi_j^s(\mathbf{r} - \mathbf{r}_j)$ is the field scattered by the jth particle to the point of observation $\mathbf{r}$.

We now observe that the field ϕ_j^e exciting the jth scatterer is the resultant of the incident field ϕ^0 and the scattered field from all other scatterers so that

$$\phi_j^e(\mathbf{r}) = \phi^0(\mathbf{r}) + \sum_{i\neq j}^{N} \phi_i^s(\mathbf{r}-\mathbf{r}_i); \quad a<|\mathbf{r} - \mathbf{r}_i|<2a, \tag{3}$$

where a is the radius of the sphere circumscribing the solid particle and the superscript e refers to the exciting field. We also assume that the transparent spheres of radius a circumscribing each particle do not interpenetrate.

The multiple scattering formulation developed here is based on the T-matrix approach, see for example, Varadan and Varadan.[15] We first expand the field quantities in terms of spherical wavefunctions

$$\begin{Bmatrix} \mathrm{Ou} \\ \mathrm{Re} \end{Bmatrix} \Psi_{lm\sigma} = \begin{Bmatrix} h_l(kr) \\ j_l(kr) \end{Bmatrix} Y_{lm\sigma}(\theta,\phi), \tag{4}$$

where h_l are the spherical Hankel functions, $Y_{lm\sigma}$ the normalized spherical harmonics, and j_l the spherical Bessel functions; $\sigma = e$ or o refers to the even or odd parity of the angular dependence, $l = 0,1,...$; $m = 0,1,...l$.

As usual, the field quantities that are regular at the origin will involve the spherical function j_l instead of h_l. We thus write

$$\phi_i^s(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \sum_{\sigma} f^i_{lm\sigma} \, \mathrm{Ou}\, \Psi^i_{lm\sigma}(\mathbf{r}-\mathbf{r}_i), \tag{5}$$

$$\phi_j^e(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \sum_{\sigma} \alpha^j_{lm\sigma} \, \mathrm{Re}\, \Psi^j_{lm\sigma}(\mathbf{r}-\mathbf{r}_j), \tag{6}$$

$$\phi^0(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \sum_{\sigma} g^0_{lm\sigma} \, \mathrm{Re}\, \Psi^j_{lm\sigma}(\mathbf{r}), \tag{7}$$

where $g^0_{lm\sigma}$ is the known incident field coefficient, $f_{lm\sigma}$ and $\alpha_{lm\sigma}$ are the unknown scattered and exciting field coefficients, respectively. The exciting and scattered fields are then related by

$$\sum_{lm\sigma} \alpha^j_{lm\sigma} \, \mathrm{Re}\, \Psi^j_{lm\sigma}(\mathbf{r}-\mathbf{r}_j) = \phi^0(\mathbf{r}) + \sum_{i\neq j}^{N} \sum_{lm\sigma} f^i_{lm\sigma} \, \mathrm{Ou}\, \Psi^i_{lm\sigma}(\mathbf{r}-\mathbf{r}_i). \tag{8}$$

The T matrix[15] can be used to relate the exciting and scattered field coefficients pertaining to a single scatterer as follows

$$f^i_{lm\sigma} = \sum_{l'm'\sigma'} T^i_{lm\sigma,l'm'\sigma'} \alpha^i_{l'm'\sigma'} . \tag{9}$$

The right- and left-hand sides of Eq. (8) refer to two different origins (the centers of the ith and jth scatterers, respectively). This is remedied by invoking the translation-addition theorem for the spherical wave functions[16] and the orthogonality of these functions can be used to extract a set of linear matrix equations for the exciting field coefficients. The details of these steps have been described in Refs. 17–19. Performing the above operations and further assuming that the scatterers are identical we obtain for $N \to \infty$,

$$\alpha^j_{lm\sigma} = a_{lme} e^{ik\hat{k}_0 \cdot \mathbf{r}_j} \delta_{m0} + (N-1) \sum_{l'm'\sigma'} \sum_{l''m''\sigma''} T_{l'm'\sigma',l''m''\sigma''} \alpha^i_{l''m''\sigma''} \times \sigma_{l'm'\sigma' lm\sigma}(\mathbf{r}_j - \mathbf{r}_i), \tag{10}$$

where $\sigma(\mathbf{r}_j - \mathbf{r}_i)$ is the translation matrix, the detailed form of which is not presented here.

If the scatterers are not identical but there is a distribution of sizes, then we may replace T in Eq. (10) by the average T matrix $\langle T \rangle$ defined as

$$\langle T \rangle = \int T(a) q(a) da, \tag{11}$$

where $q(a)$ is a function specifying the size distribution of particles assuming that they are all of the same shape. In order to further refine the calculations one may want to perform the average in Eq. (11) only after introducing the configurational average since the nearest distance between particles is a function of their size. To a first approximation, we can use $\langle T \rangle$ in Eq. (10) and use $\bar{a}$ the mean radius to describe the particle size.

II. CONFIGURATIONAL AVERAGE AND THE PAIR CORRELATION FUNCTION

For a system with a large number of scatterers, it is more meaningful to study the effective propagation characteristics in the medium rather than the details of the multiple scattering processes that take place. Thus a configurational average is performed in Eq. (10) over the positions of all particles except the jth which is assumed to be held fixed. We thus have

$$\langle \alpha^{i}_{lm\sigma} \rangle_j = a_{lm\sigma} e^{ikk_0 \cdot r_j} \delta_{m0} + (N-1) \sum_{l'm'\sigma'} \sum_{l''m''\sigma''} T_{l'm'\sigma',l''m''\sigma''} \times \int \langle \alpha^{i}_{l''m''\sigma''} \rangle_{ij} \sigma_{l'm'\sigma',lm\sigma}(\mathbf{r}_j - \mathbf{r}_i) p(\mathbf{r}_i | \mathbf{r}_j) d\mathbf{r}_j, \tag{12}$$

where $\langle\ \rangle_i$ denotes the configuration average with the ith scatterer held fixed and $p(\mathbf{r}_i|\mathbf{r}_j)$ is the conditional probability distribution function.

Equation (12) is a hierarchy which when iterated will involve higher conditional probability distribution functions. The hierarchy is truncated by invoking the quasi-crystalline approximation (QCA) first suggested by Lax[20] which works surprisingly well for a wide range of concentrations. According to the QCA

$$\langle \alpha^{i}_{l''m''\sigma''} \rangle_{ij} = \langle \alpha^{i}_{l''m''\sigma''} \rangle_{i}. \tag{13}$$

To study the coherent or average field in the effective medium, we assume that the average field is a plane wave propagating in the same direction $\hat{k}_0 = \hat{z}$ as the original plane wave but with a complex propagation constant $K = K_1 + iK_2$ which is frequency dependent. The real part K_1 is related to the phase velocity and K_2, the imaginary part is proportional to the attenuation constant. Thus

$$\langle \alpha^{i}_{lm\sigma} \rangle_i = X_{lm\sigma} e^{iK\hat{z} \cdot \mathbf{r}_i}. \tag{14}$$

In order to complete the integration in Eq. (12), the joint probability function must be specified. It is convenient to write

$$P(\mathbf{r}_i|\mathbf{r}_j) = \begin{cases} g(x)/V, & x > 1 \\ 0, & x < 1, \end{cases} \tag{15}$$

where we have assumed that the particles are impenetrable and that for a translationally invariant system, $P(\mathbf{r}_i|\mathbf{r}_j)$ depends only on $|\mathbf{r}_i - \mathbf{r}_j| = 2ax$ where $2a$ is the hard core radius or the minimum distance between particles, each of radius a. In the statistical mechanics literature $g(x)$ is known as the radial distribution function.

Equations (13)–(15) are substituted in Eq. (12) and the extinction theorem can be invoked to cancel the incident wave term on the right-hand side of Eq. (12) with part of the second term (refer to Ref. 18 for details). The resulting equation is

$$X_{lo\epsilon} = \sum_{l'} \sum_{l''} T_{l'o\epsilon,l''o\epsilon} X_{l''o\epsilon} \times \sum_{\lambda = |l'-l|}^{l+l'} i^{l'-l}(2l+1) a(0,l'|0,l|\lambda) JH_\lambda \times (\{6c/[(ka)^2 - (Ka)^2]\} + 24cI_\lambda), \tag{16}$$

where

$$JH_\lambda = 2ka j_\lambda(2Ka) h'_\lambda(2ka) - 2Ka h_\lambda(2ka) j_\lambda(2Ka), \tag{17}$$

$$a(0,l'|0,l|\lambda) = (2\lambda + 1) \begin{bmatrix} l & l & \lambda \\ 0 & 0 & 0 \end{bmatrix}^2, \tag{18}$$

where $\begin{bmatrix} l & l & \lambda \\ 0 & 0 & 0 \end{bmatrix}$ is the Wigner 3-j symbol.[16]

$$I_\lambda = \int_{x=1}^{\infty} x^2 [g(x) - 1] j_\lambda(2Kax) h_\lambda(2kax) dx, \tag{19}$$

and $c = (4\pi/3) a^3 N/V$ is the fractional volume occupied by the particles.

Equation (16) is a system of linear simultaneous equations for the coefficients X_l. For a nontrivial solution of the coherent field we must require the determinant of the coefficient matrix to vanish. This is the required dispersion equation which can be solved for the effective propagation constant K as a function of $k = \omega/c_f$ and c. The determination of K from Eq. (16) is necessarily numerical except in the long wavelength limit. This will be discussed in the next section. We now proceed to discuss the forms of $g(x)$ that will be used to numerically evaluate I_λ.

Several models of $g(x)$ are available. For uncorrelated, impenetrable particles

$$g(x) = \begin{cases} 1/(1-c), & x > 1 \\ 0, & x < 1. \end{cases} \tag{20}$$

This approximation for $g(x)$ refers to Talbot and Willis,[21] known as the well-stirred approximation (WSA), and is expected to be valid for low values of c, and as discussed by Bringi *et al.*,[9] fails at $c > 0.125$. For concentrations $c < 0.125$, there is an analytical form for $g(x)$ due to Matern.[22] Twersky[8] has used a virial expansion to obtain $g(x)$ shown as

$$g(x) = \begin{cases} 0, & x < 1 \\ 1 + 8c(1 - \frac{3}{4}x + \frac{1}{16}x^3), & 1 < x < 2 \\ 1, & x > 2 \end{cases} \tag{21}$$

which is valid at low concentrations.

Improved models of the pair correlation function valid for concentrations up to 40% are the Percus-Yevick approximation (PYA) and the Hypernetted-Chain approximation (HNC). The Percus-Yevick model[23] has been solved analytically by Wertheim[24] for the case of hard impenetrable particles. It is expected to be somewhat better than the HNC.[25] One of the defects of the PYA is that the two equations that can be derived for the pressure P in a fluid containing "hard" particles lead to different answers when the PYA for $g(x)$ are substituted in them (these equations are derived by Percus[23]). Rowlinson[26] remedied this by assuming that the direct correlation function which is the short range part of the correlation function is a linear combination of the ones resulting from the PYA and HNC models. They were combined with an adjustable parameter ϕ and the two pressure equations were solved simultaneously for P and ϕ. This is called the self-consistent approximation (SCA) and it is valid for higher concentrations than the PYA and HNC models.

III. RAYLEIGH LIMIT SOLUTION

The dispersion relation derived from Eq. (16) can be solved in detail to predict the attenuation features and wave speeds of coherent acoustic waves in the two-phase medium. Although the system of equations requires a numerical approach to yield solutions for higher values of frequency, analytical results can be obtained for low-frequency approximations. Including the effects of correlation between particles it is seen that an attenuation factor is obtained even in the low-frequency approximation. Analytical results are seen to mainly depend on the form of the correlation assumed. Using, for example, the $g(x)$ given by the virial series, Eq. (21), and for leading order in (ka), dispersion equations can be derived from Eq. (16) for all the three types of scatterers, namely rigid, fluid, and elastic spheres. We get for the real and imaginary parts of the effective wavenumber

$$(K_1/k)^2 = (1+\alpha c)(1-\beta c)/(1+2\beta c) \tag{22}$$

and

$$2(K_2/K_1) = (ka)^3(1 - 8c + 34c^2) \times [\alpha^2/(1+\alpha c) + 3\beta^2/(1+2\beta c)(1-\beta c)]/3, \tag{23}$$

where

$$\alpha = \left\{ \begin{array}{ll} -1 & \text{rigid} \\ \dfrac{\rho c_p^2}{\rho_f c_f^2} - 1 & \text{fluid sphere,} \\ \left[\left(\dfrac{\rho c_p^2}{\rho_f c_f^2}\right) - \dfrac{4\rho c_s^2}{3\rho_f c_f^2}\right]^{-1} - 1 & \text{elastic} \end{array} \right\} G(\bar{a}), \tag{24}$$

$$\beta = \left\{ \begin{array}{ll} -\frac{1}{2} & \text{rigid} \\ \dfrac{\rho_f - \rho}{\rho_f + 2\rho} & \text{fluid sphere .} \\ \dfrac{\rho_f - \rho}{\rho_f + 2\rho} & \text{elastic} \end{array} \right\} G(\bar{a}),$$

For a uniform size distribution $G(\bar{a}) = 1$, while for the Gaussian size distribution $G(\bar{a})$ is given by

$$\begin{aligned} G(\bar{a}) = {} & 2\left(1 + \frac{\bar{a}^2}{2m^2}\right)\left(\frac{m}{\bar{a}}\right)\exp\left(\frac{-\bar{a}^2}{2m^2}\right)\Big/(2\pi)^{1/2} \\ & + 1.5\left[2 - \exp\left(-\frac{\bar{a}}{\sqrt{2}m}\right)\right]\left(\frac{m}{\bar{a}}\right)^2 \\ & + 3\left(\frac{m}{\bar{a}}\right)\exp\left(-\frac{\bar{a}^2}{2m^2}\right)\Big/(2\pi)^{1/2} \\ & + 0.5\left[1 + \operatorname{erf}\left(\frac{\bar{a}}{\sqrt{2}m}\right)\right], \end{aligned} \tag{25}$$

where m is the standard deviation. When the concentration is higher than about 5% the term $(1 - 8c + 34c^2)$ is replaced by $(1-c)^4/(1+2c)^2$ in Eq. (23) to correct for the higher concentration using the Ornstein–Zernike equation[8] for $g(x)$. At low frequencies the attenuation is due to multiple scattering alone since the scatterers are assumed lossless. However, losses due to other factors such as viscosity, friction, etc., may dominate the scattering losses (which are proportional to k^3a^3) at low frequencies. At higher frequencies, the multiple scattering losses increase significantly and may dominate the viscous or frictional losses. Methods of introducing these additional loss mechanisms into the current theory are under investigation by the authors.

IV. PHASE VELOCITY AND ATTENUATION AT HIGHER FREQUENCIES

The analytical expressions for wave speed and attenuation factor as obtained above could be derived only for very low values of ka as higher approximations would lead to unwieldy expressions. A quantitative estimate of the multiple scattering process at resonant and higher frequencies can be obtained by numerically solving the dispersion equation, Eq. (16). In fact, detailed numerical results can be obtained for rigid, fluid, or elastic scatterers of different geometries as the only additional input parameter in such cases is the appropriate T matrix. The computational scheme has been described previously and will not be repeated here.[27,28]

Essentially, the solution of the dispersion equation derived from Eq. (16) valid at higher frequencies involves an iterative procedure for determining the dominant root in the complex plane in terms of ka. Although the algorithm used

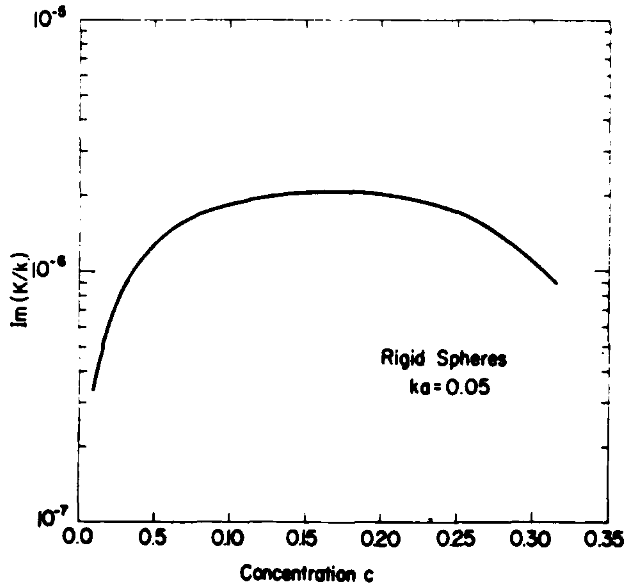


FIG. 1. Coherent wave attenuation versus concentration for $ka = 0.05$ for rigid spheres embedded in water.

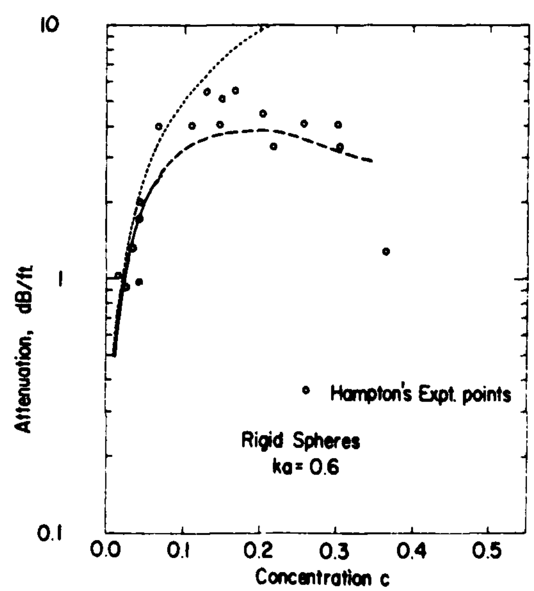


FIG. 2. Attenuation in dB/ft versus concentration for rigid spheres for $ka = 0.6$.

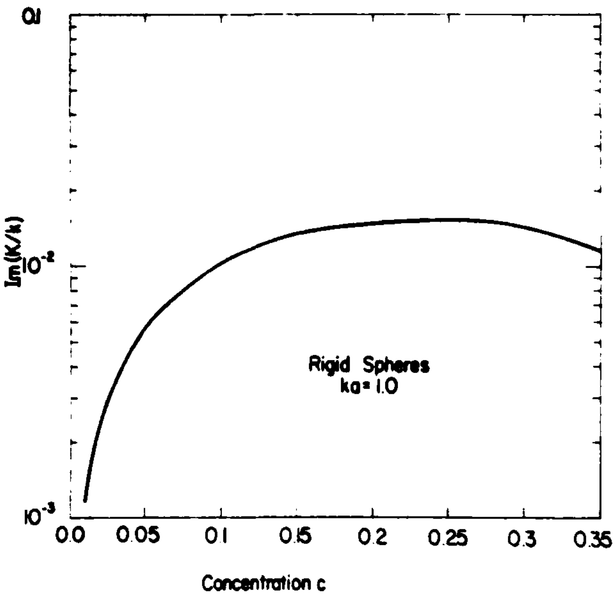


FIG. 3. Attenuation coefficient versus concentration for rigid spheres in water for $ka = 1.0$.

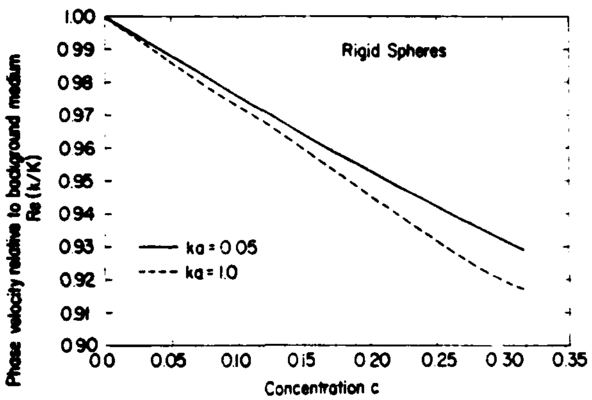


FIG. 4. Relative phase velocity versus concentration for $ka = 0.05$ and 1.0.

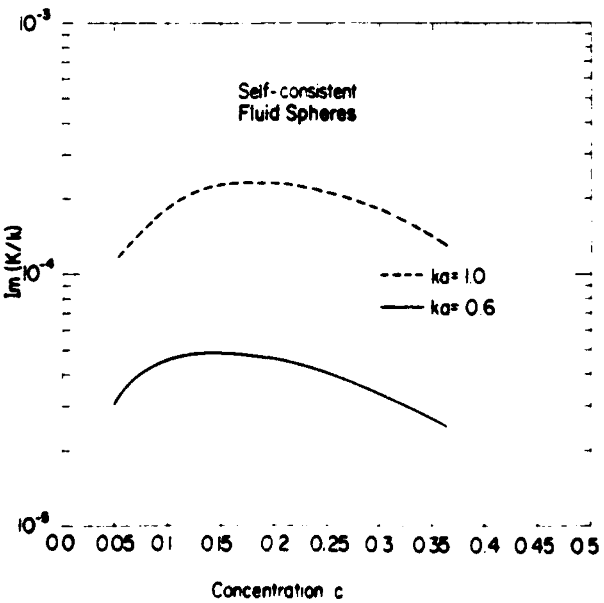


FIG. 5. Attenuation coefficient versus concentration for spherical "red blood cells" in a plasma medium for $ka = 0.6$ and 1.0 using SCA.

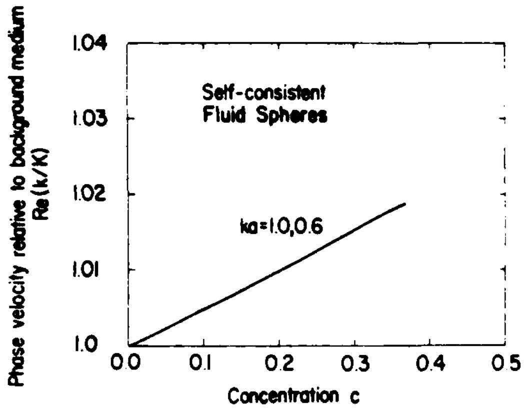


FIG. 6. Relative phase velocity versus concentration for spherical "red blood cells" in a plasma medium for $ka = 0.6$ and 1.0 using SCA.

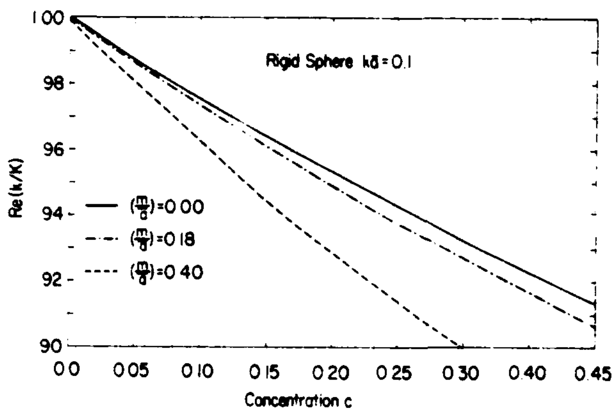


FIG. 7. Relative phase velocity versus concentration for rigid spheres with Gaussian size distributions.

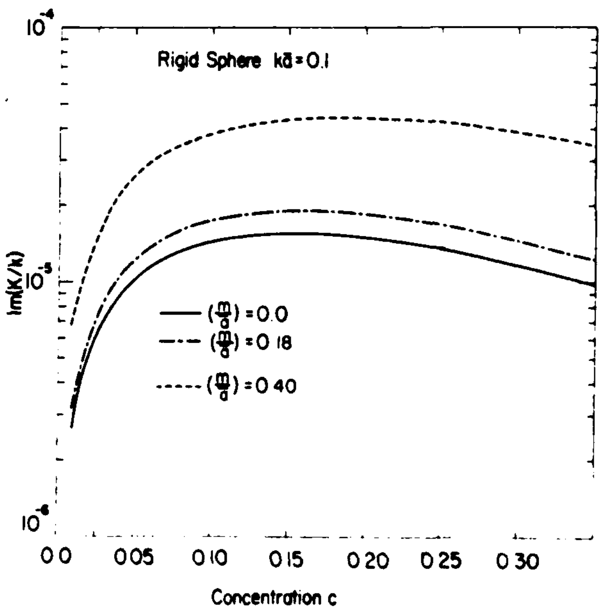


FIG. 8. Attenuation coefficient versus concentration for rigid spheres with Gaussian size distributions.

appears to be insensitive to considerable deviations from the true solution, good initial values are provided by the low-frequency results of the previous section. As expected, the order of the coefficient matrix that has to be retained in the amplitude equations for a convergent solution is directly proportional to the frequency. The same is also true with respect to the size of T matrix.

In Fig. 1 we show the imaginary part of the propagation constant K/k for rigid spheres immersed in water with $ka = 0.05$ where k is the wavenumber of the incident acoustic wave in water. The curve depicts the correct functional form for K_2 vs c and shows a broad maximum in the range $0.15 < c < 0.20$. At higher concentrations, the attenuation decreases implying that the random medium appears more homogeneous with "restricted" randomness. In the limit full packing ($c \simeq 0.64$ for spheres), the attenuation should attain a fixed value consistent with a bulk rigid medium. The form for $g(x)$ in the limit or full packing is not known, hence computations could not be performed.

In Fig. 2 we show a similar plot of coherent attenuation versus concentration for rigid spheres in water for comparison with measurements conducted by Hampton.[29] Note that the ordinate scale is in dB per foot where dB/ft $= 1109(K_2/k)$. The value of ka is chosen to be 0.6 so as to yield computed attenuation values in the range observed by Hampton (his experimental points are shown by circles in Fig. 2). His experiments involved clay particles in a polydisperse distributions embedded in water with maximum ka values much smaller than 0.6. Computation at such small ka values yield attenuations which are much smaller than observed since K_2 is of $O(k^3a^3)$. We conclude that the experimentally observed attenuation is due to loss mechanisms other than multiple scattering. In any case the present computations agree favorably with experimental values at $ka = 0.6$ showing the observed functional dependence with increasing concentration. Note that the computed attenuation is due to multiple scattering losses only. For $c > 0.05$, the single scattering approximation significantly overestimates the attenuation, increasing without limit as c approaches dense packing. The self-consistent form for $g(x)$ appears most suitable for the full

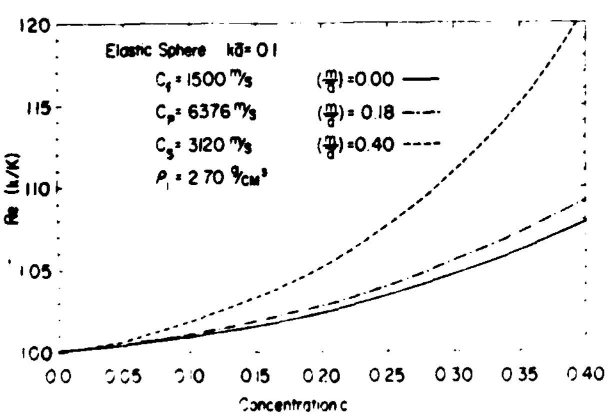


FIG. 9. Relative phase velocity versus concentration for elastic spheres with Gaussian size distributions.

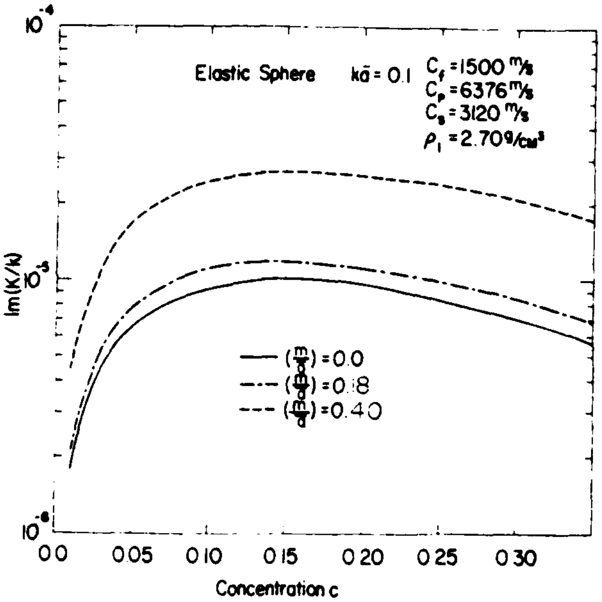


FIG. 10. Attenuation coefficient versus concentration for elastic spheres with Gaussian size distributions.

range of concentrations. In Fig. 3 we again show K_2/k versus concentration for $ka = 1.0$. Figure 4 shows the phase velocity in the bulk medium relative to the phase velocity of the background medium in which the scatterers are embedded. At higher concentrations there is evidence of dispersion as seen by the separation of the $ka = 0.5$ and $ka = 1.0$ curves. The bulk or composite medium supports "slow" acoustic waves when the scatterers are rigid.

Computations were also performed for fluid and elastic scatterers with the only change in the computation occurring as a result of the appropriate T matrix[30,31] in Eq. (16). The acoustic parameters of the fluid scatterers and the surrounding fluid medium were chosen to be $\rho = 1.092$ g cm^{-3}, $c_p = 1.64 \times 10^5$ cm/s, and $\rho_f = 0.021$, $c_f = 1.55 \times 10^5$, respectively, where ρ is the density and c is the sound velocity.[32] These parameters are appropriate for red blood cells (RBC) in isotonic plasma although a spherical model for the RBC is an approximation. Figure 5 shows the computed attenuation coefficient as a function of concentration for $ka = 0.6$ and $ka = 1.0$ using the self-consistent pair-correlation function. Note that the computed attenuation is due to multiple scattering. In Fig. 6 we show the relative phase velocity versus c for $ka = 0.6$ and 1.0 using the self-consistent model. The bulk medium now appears nondispersive and supports "fast waves." In Fig. 7, the phase velocity versus concentration c is shown for Gaussian size distribution ($m/\bar{a} = 0.18$ and 0.4) while the corresponding coherent attenuation curves are shown in Fig. 8. In Figs. 9 and 10, the phase velocity and coherent attenuation for elastic scatterers with Gaussian and uniform size distributions are depicted. The properties of the elastic scatterers are taken as $c_p/c_s = 2.04$, $\rho/\rho_f = 2.7$.

appears to be insensitive to considerable deviations from the true solution, good initial values are provided by the low-frequency results of the previous section. As expected, the order of the coefficient matrix that has to be retained in the amplitude equations for a convergent solution is directly proportional to the frequency. The same is also true with respect to the size of T matrix.

In Fig. 1 we show the imaginary part of the propagation constant K/k for rigid spheres immersed in water with $ka = 0.05$ where k is the wavenumber of the incident acoustic wave in water. The curve depicts the correct functional form for K_2 vs c and shows a broad maximum in the range $0.15 < c < 0.20$. At higher concentrations, the attenuation decreases implying that the random medium appears more homogeneous with "restricted" randomness. In the limit full packing ($c \simeq 0.64$ for spheres), the attenuation should attain a fixed value consistent with a bulk rigid medium. The form for $g(x)$ in the limit or full packing is not known, hence computations could not be performed.

In Fig. 2 we show a similar plot of coherent attenuation versus concentration for rigid spheres in water for comparison with measurements conducted by Hampton.[29] Note that the ordinate scale is in dB per foot where $\mathrm{dB/ft} = 1109(K_2/k)$. The value of ka is chosen to be 0.6 so as to yield computed attenuation values in the range observed by Hampton (his experimental points are shown by circles in Fig. 2). His experiments involved clay particles in a polydisperse distributions embedded in water with maximum ka values much smaller than 0.6. Computation at such small ka values yield attenuations which are much smaller than observed since K_2 is of $O(k^3a^3)$. We conclude that the experimentally observed attenuation is due to loss mechanisms other than multiple scattering. In any case the present computations agree favorably with experimental values at $ka = 0.6$ showing the observed functional dependence with increasing concentration. Note that the computed attenuation is due to multiple scattering losses only. For $c > 0.05$, the single scattering approximation significantly overestimates the attenuation, increasing without limit as c approaches dense packing. The self-consistent form for $g(x)$ appears most suitable for the full range of concentrations. In Fig. 3 we again show K_2/k versus concentration for $ka = 1.0$. Figure 4 shows the phase velocity in the bulk medium relative to the phase velocity of the background medium in which the scatterers are embedded. At higher concentrations there is evidence of dispersion as seen by the separation of the $ka = 0.5$ and $ka = 1.0$ curves. The bulk or composite medium supports "slow" acoustic waves when the scatterers are rigid.

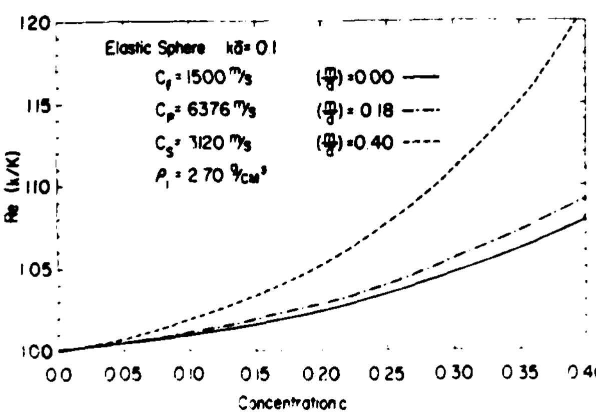
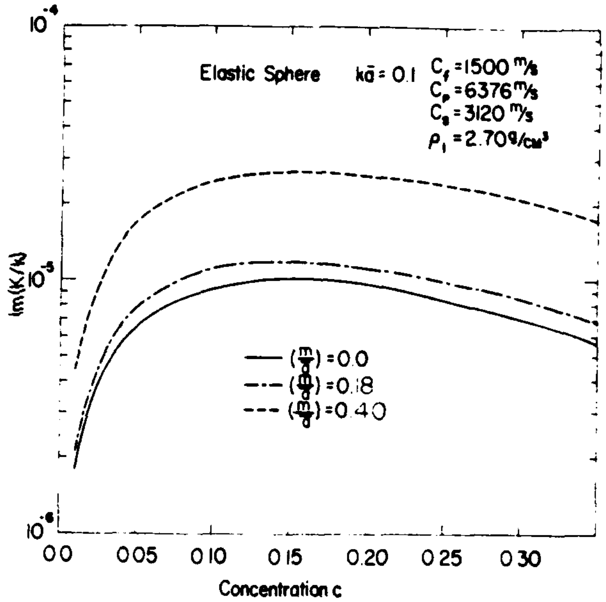


FIG. 9. Relative phase velocity versus concentration for elastic spheres with Gaussian size distributions.



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V. CONCLUSIONS

This paper has presented a T-matrix formulation of multiple scattering of acoustic waves in a two-phase random medium. The dispersion equation, which characterizes the propagation of coherent waves in the effective medium, is derived by including the effects of pair-correlation. Explicit expressions for the attenuation coefficient in the Rayleigh limit are derived using the virial series expansion of the radial distribution function. Computations are performed for rigid, fluid, and elastic spheres embedded in a fluid background medium using a number of models for the radial distribution function. At high concentrations the self-consistent model appears to give superior results based on a comparison with experimental observations. The theory and computational procedure developed in this paper should prove useful in dealing with acoustic wave propagation through ocean-bottom sediments, mixtures, and suspensions and composite biological tissues.

ACKNOWLEDGMENTS

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Wave propagation in pair-correlated particles of varying size suspended in water

by

Y. Ma, V.K. Varadan and V.V. Varadan
Wave Propagation Group
Department of Engineering Mechanics
The Ohio State University, Columbus, Ohio 43210

ABSTRACT

Acoustic waves propagating through suspended materials with a considerable concentration ($1\%<w<40\%$) in the deep ocean environment is examined. The particles in suspension have a size distribution and their relative positions are described by the pair-correlation function. Different equations representing the pair-correlation function are investigated. The characteristics of the multiple scattered waves are presented as the dispersion of the phase velocity specified by the real part of the effective wave number K and the attenuation in wave intensity shown as the loss tangent.

The dependence of the wave propagation on the size distribution and concentration of particles is also discussed in the low frequency range.

INTRODUCTION

An acoustic technique has been proposed to investigate the wave propagation through the suspended sedimentary particles in a water mass[1]. The attenuation of acoustic waves by suspended materials in a fluid medium has been investigated extensively[2-4] and the experimental attenuation coefficients were found to be compatible with those derived from theory for low concentration cases. However, when the concentration becomes higher the acoustic transmission and reflection will be affected differently due to multiple scattering. The single scattering theory[5] which is suitable for a sparse distribution of scatterers (suspended particles) is no longer valid for a dense distribution of scatterers since the higher order statistics known as the pair-correlation[6] will be required in the multiple scattering analysis. In other words, in a medium containing a large number of scatterers the position of one scatterer is constrained by the other therefore the scattered field will be affected by such a crowding characterized by a correlation function among scatterers.

The present study examines acoustic wave propagation through a considerable concentration of sedimentary particles suspended in water. The quasicrystalline[7] assumption will be used to truncate the hierarchy equations (Foldy-Lax hierarchy[5]) so that only the pair-correlation between two particles is considered. As can be seen the pair-correlation function is essential in solving the coherent field. Therefore, the hole correction[8], the virial expansion[9] and the Percus-Yevick[10] equations representing the pair-correlation function are discussed. However, it can be shown that all those equations will yield the similar results for moderate concentrations of scatterers in the low frequency range.

For practical purposes, a simpler form of the pair-correlation function for elastic scatterers is given instead of the complex Percus-Yevick equation. The scatterers with and without size distributions are both treated in this study to judge their effects on the wave propagation characteristics. The results presented as the dispersion of the phase velocity as well as the attenuation against the concentration are particularly in the low frequency range.

STATEMENT OF THE PROBLEM

Consider a slab containing a large number of scatterers. A time-harmonic plane wave of sound is normally incident upon the slab (see Figure 1) along the negative z direction with the time dependence $e^{+i\omega t}$. The incident wave with a unit amplitude can thus be expressed as

$$\psi(\vec{R}) = e^{ikz}$$

where k = wave number.

The scatterers are suspended particles modelled as elastic spheres whose properties are shown in Table 1. The surrounding acoustic medium (covering slab) is water with the properties also shown in Table 1. In the absence of scatterers from the medium, $\psi(\vec{R})$ will then satisfy the wave equation

$$\nabla^2\psi + k^2\psi = 0 \tag{1}$$

However, the average field (coherent field) $\langle\psi\rangle$[11] for any wave in a medium containing random particles can generally be expressed by assuming it satisfies the following wave equation specified by an effective wave number[5] K (sometimes called propagation constant or bulk parameter), i.e.,

$$\nabla^2\langle\psi\rangle + K^2\langle\psi\rangle = 0$$

For low concentrations (the number of scatterers in the medium is small and the distances between scatterers are large compared to the incident wave length) the effective wave number K is well known as[12]. hen including the size distribution,

$$K^2 = [k+\frac{2\pi\rho}{k}\,\overline{(f(a,\pi)} + \overline{f(a,o))}]\,[k+\frac{2\pi\rho}{k}\,\overline{(f(a,\pi)} - \overline{f(a,o))}], \tag{2}$$

where ρ = number density (number of scatterers per volume)

$$\overline{f(a,\pi)} = \int_o^\infty f(a,\pi)\, q(a)da$$

$$\overline{f(a,o)} = \int_o^\infty f(a,o)\, q(a)da$$

$q(a)$ = the size distribution function

$f(a,\pi)$ = the forward scattering function[1]

$f(a,o)$ = the backscattering function[1]

Lloyd and Berry[13] modified the above equation using the multiple scattering treatment as follows

$$K^2 = k^2+4\pi\rho f(a,\pi) + \frac{4\pi^2\rho^2}{k^2}\,[-f^2(a,\pi)+f^2(a,o) - \int_o^\pi \frac{1}{\sin\frac{1}{2}\theta}\,\frac{d}{d\theta}\,f^2(\theta)d\theta], \tag{3}$$

where

$$f(\theta) = \sum_{n=o}^{\infty} \frac{(2n+1)i(-1)^n}{1+iC_n}\,P_n(\cos\theta).$$

Another frequently used equation for the effective wave number K is given as,[15]

$$K^2 = k^2+4\pi\rho f(a,\pi) + (4\pi\rho f(a,\pi))^2 \int_o^\infty e^{ikR}\,\frac{\sin kR}{k}\,g(R)dR, \tag{4}$$

where $g(R)$ is the radial distribution function. Results using Eqs. (2), (3) and (4) will be examined in this paper.

Equation (2) can be further reduced to the following form provided $\rho<<1$,

$$K^2 = k^2 + 4\pi\rho\overline{f(a,\pi)} + O\ (\rho^2) \tag{5}$$

in which O ( ) represents the order of ( ). For small scatterers and low concentrations a common form used by many investigators is obtained by expanding the square root of Eq. (5) as

$$K = k + 2\pi\rho\overline{f(a,\pi)}/k \tag{6}$$

One sees from Eq. (6) that K is actually a complex number due to the fact that $\overline{f(a,\pi)}$ is complex. The real and imaginary parts of K are found to be

$$\begin{aligned} K_R &= k + 2\pi\rho\overline{f_R(a,\pi)}/k \\ K_I &= 2\pi\rho\overline{f_I(a,\pi)}/k, \end{aligned} \tag{7}$$

where the subscripts R and I denote real and imaginary respectively.

By using the forward scattering theorem which is

$$\sigma_s = \frac{4\pi}{k}\overline{f_I(a,\pi)}, \tag{8}$$

the imagninary part of K (also known as the attenuation constant) can thus be related to the total scattering cross section σ_s as

$$2K_I = \rho\sigma_s + (\rho\sigma_a) \tag{9}$$

In Eq. (9) σ_a is the absorption cross section only if the scatterers are absorbent (the absorption mechanisms can be introduced by using complex internal wave numbers in the scattering functions $\overline{f(a,\pi)}$ and $\overline{f(a,o)}$ or it will not appear. The absorption effect for sedimentary particles is small and can be neglected.[1]

Strictly speaking, Eqs. (5), (6), (7) and (9) can only be applied to the low concentration cases. Each of the sparsely distributed scatterers under such circumstance can be treated as independent scatterers, i.e., uncorrelated and without interference. Therefore the total scattering is just the sum of the scattering from each scatterer which is the basis of the single scattering theory. A question will arise as to how large a concentration will make the single scattering invalid? The answer depends on the type of scatterers being treated.[1] However, a recent experimental study by Ishimaru et.al.[15] shows that when the concentration is greater than about 0.1%, the experimental attenuation constant departs markedly from that calculated by single scattering theory.

The term concentration used in this paper is defined as the volumetric percentage which is the ratio between the total volume of scatterers and the volume of the embedding medium. Since the spherical scatterers are modelled as the suspended particles the concentration w is thus represented as

$$w = 4\pi\rho\bar{a}/3,$$

where a is the radius of the sphere and the overbar denotes the average quantity.

To examine the effective wave number K for a dense distribution of scatterers (high concentration) one needs to recall the Foldy-Lax hierarchy for the coherent field holding more than one scatterers fixed. Fortunately, the problem can be simplified without going into higher order statistics by using the quasicrystalline approximation to truncate the equations in order that only the pair statistics between two scatterers is required in evaluating the coherent field. Such a treatment is useful in which an explicit approximation for the pair statistics, expressed as the pair distribution function g(R) can be introduced to integrate over the total volume to obtain the exciting coherent field. The second equation in the Foldy-Lax hierarchy using the quasicrystalline approximation can generally be written as

$$<\psi^j(\vec{R}_j)>_j = \psi_{inc}(\vec{R}_j) + \rho \int g(\vec{R}_i-\vec{R}_j)\overline{f(a,\theta)} <\psi^i(\vec{R}_i)>_i \, d\vec{R}_i \; E(\vec{R}_i-\vec{R}_j) \quad (10)$$

where $<\psi^i(\vec{R}_j)>_j$ = the average exciting field of the jth scatterers with its position held fixed

$\psi_{inc}(\vec{R}_j)$ = the incident field at $\vec{R}_j$

$\overline{f(a,\theta)} = \int_0^\infty f(a,\theta)\, q(a)da$

$<\psi^i(\vec{R}_i)>_i$ = the average exciting field of the ith scatterer with its position held fixed

$E(\vec{R}_i-\vec{R}_j)$ = propagation function of the scattered waves

$d\vec{R}_i = 4\pi R^2 dR$ for the radially symmetric scatterers

SOLUTION FOR RADIALLY SYMMETRIC SCATTERERS

The effective wave number K for a dense distribution of pair-correlated scatterers can be obtained using Eq. (10) as follows[9]

$$K^2 = k^2 + \frac{4\pi\rho}{k} \sum_{n=o}^{\infty} A_n \eta^{2n} \tag{11}$$

where $\eta = K/k$

$$A_n = k\overline{f_n(a,\pi)}\,[1+\sum_{m=o}^{\infty} i\, A_n (\eta)^{-n+m} H_{nm}]$$

$$\overline{f_n(a,\pi)} = \int_o^{\infty} f_n(a,\pi)\, q(a)da = \overline{f}_n$$

$$f_n(a,\pi) = \frac{(2n+1)i}{1+iC_n}$$

$$H_{nm} = \sum_{|n-m|}^{|n+m|} dm \begin{pmatrix} o & o \\ n & m \end{pmatrix} \left[\frac{4\pi\rho i}{k^3} \frac{\eta^{m+n} - \eta^{m}}{\eta^2 - 1} + N_m\right]$$

$$N_m = 4\pi\rho \int_o^{\infty} [g(R)-1] j_m(KR) h_m^2 (kR) R^2 dR \tag{12}$$

$$= N_m^1 + i\, N_m^2$$

$j_m(KR)$ = spherical Bessel function

$h_m^2(kR)$ = spherical Hankel function of the second kind

$g(R)$ = pair-correlation function

$dm\begin{pmatrix} o & o \\ n & m \end{pmatrix}$ = Lengendre expansion for

$$P_n(x)P_m(x) = \sum_{q=o}^{n+m} dq\begin{pmatrix} o & o \\ n & m \end{pmatrix} P_q(x)$$

$P_n(x)$, $P_m(x)$, $P_q(x)$ = Lengendre polynomials

$dq\binom{o;o}{n;m}$ has the following properties

$$dq\binom{o;o}{n;m} = dq\binom{o;o}{m;n}$$

$$d_o\binom{o;o}{n;m} + d_1\binom{o;o}{n;m} + \ldots + dq\binom{o;o}{n;m} = 1 \quad , \ q = m+n$$

$$d_{odd}\binom{o;o}{m;n} = 0 \quad , \quad m+n = \text{even}$$

$$d_{even}\binom{o;o}{m;n} = 0 \quad , \quad m+n = \text{odd}$$

For monopole and dipole cases

$$d_o\binom{o;o}{o;1} = 0 \quad , \quad d_1\binom{o;o}{o;1} = 1$$

For the low frequency case, only the monopole (each scatterer, i.e. sphere, is compressed and expanded by the incident condensations and rare factions and a spherical wave is thus radiated - this monopole type radiation is dominated by the compressibility of the scatterer and is independent of directions) and dipole (the scatterer's inertia causes the scatterers to have a motion which is equivalent to the surrounding medium being at rest and the scatterer being in oscillation - this dipole type reradiation is affected by the density of the scatterer and depends on the scattering angle θ) terms are important in doing the computation. Therefore only two terms in Eq. (11) are required and a closed form solution can be obtained. However, more terms are necessary in the high frequency range which make the calculation for K a tedious job and only a numerical approach is possible at the present time. This can be done by the computer for a wide frequency range by selecting a suitable number terms to achieve convergence. The acoustic scattering analysis for suspended particles in the high frequency range will be discussed in a later paper.

Now one can compare Eq. (11) with Eq. (5) to see the difference. To study the wave propagation through a sparse distribution of scatterers does not involve the iterative solution for K which is instead required for a dense distribution of pair-correlated scatterers. The effective wave number K in a medium containing scatterers of low concentrations is essentially a slight perturbation of Eq. (11) with $\eta \doteq 1$. One also sees from Eq. (12) that in order to obtain N_m the pair-correlation function (or the radial distribution function for the radially symmetric scatterers) g(R) needs to be specified first. If there is a closed form expression for g(R), the computation can be made much easier depending on how simple the form is.

RADIAL DISTRIBUTION FUNCTION

Several radial distribution functions (for the radial symmetric scatterers the pair correlation function g(R) is called the radial distribution function g(R))[14] have been applied successfully to analyze the multiple scattering of waves by a random distribution of pair-correlated scatterers. The commonly used hole correction equation has been proved to be a poor approximation for appreciable concentrations. Twersky[9] used the virial expansion equation which is an iterative solution for the Percus-Yevick equation for impenetrable scatterers to obtain the effective wave number K for the low frequency range and moderate concentrations.

To make the calculation easier the following form is proposed to avoid a more complex form in the expression of the Percus-Yevick equation. That is

$$g(R) = \begin{cases} 0 \quad , & R < b' \\ 1 + A'(w)e^{-B'(w)(R-b')}\cos C'(w)(R-b') \quad , & R \geq b' \end{cases} \tag{13}$$

where A'(w), B'(w) and C'(w) are functions of the concentration w and b' the separation distance (exclusion length) between two scatterers. For different scatterers with different distributions A'(w), B'(w) and C'(w) have different forms and are decided by the distribution measurements from the field data. Generally speaking, the magnitudes of A'(w) and C'(w) increases with the increasing concentration and that of B'(w) instead of increasing decreases as the concentration increases.

The idea in establishing Eq. (13) for g(R) is from the experimental histogram of the radial distribution function for the manganese nodule fields[16] (nodules are elastic materials) and the Percus-Yevick equation.

One sees that $g(R)$ has the trend of decaying and oscillating about the value one when the radial distance R is greater than the exclusion length b(=2a). The asymptotic value of $g(R)$ is one when R is generally three or four times larger than the exclusion length (see Figure 2).

For a moderate concentration, e.g. w=0.1, a comparison of $g(R)$ using different expressions is presented as Figure 3. As can be seen from the figure, (in the range of $0 \leq R \leq 2b$) the curves representing the Percus-Yevick equation, the virial expansion and the proposed radial distribution function which has the following form are very close to each other.

$$g(R) = \begin{cases} 0 \quad , \quad R < b \\ 1+0.3e^{-\pi(R-b)/b}\cos \pi(R-b)/2b \quad , \quad R \geq b \end{cases} \quad \text{(for } w = 0.1)$$

As a matter of fact, the virial expansion of $g(R)$ is the simplest one in doing the calculation with a moderate concentration.

To calculate the effective wave number K, the equation of N_m(Eq.(12)) which involves the radial distribution function $g(R)$ needs to be obtained first. In general, N_m can be expressed as the following integral.

$$N_m = 4\pi\rho F(K,k) \int_0^\infty [g(R)-1]R^\alpha dR \tag{14}$$

where $F(K,k)$ is the function of the effective wave number K and the incident wave number k. Eq. (14) is generated due to the ascending series expansion of $j_m(KR)$ and $h_m^{(2)}(kR)$ in Eq. (12) as follows

$$j_m(X) = 2^m \quad X^m \sum_{\nu=0}^{\infty} \frac{(-1)^\nu (m+\nu)\ !\ X^{2\nu}}{\nu!(2m+2\nu+1)!}$$

$$h_m{}^2\ (X) = j_m(X) - i\ y_m(X) \tag{15}$$

$$y_m(X) = \frac{1}{2^m X^{m+1}} \sum_{\nu=o}^{\infty} \frac{\Gamma(2m-2\nu+1)X^{2\nu}}{m!\Gamma(m-\nu+1)}$$

where Γ ( ) is the Gamma function.

In general, the integral

$$I^\alpha = \int_o^\infty [g(R)-1]R^\alpha dR \tag{16}$$

can be expressed as a recurssion formular which is useful in obtaining the iterative solution using the computer. A recurssion formular of Eq. (16) using Eq. (13) for g(R) can be expressed as

$$I^\alpha = T^\alpha - \frac{(b')^{\alpha+1}}{\alpha+1},$$

where

$$T^\alpha = \int_{b'}^\infty e^{-B'(R-b')} \cos C'(R-b')R^\alpha dR$$

$$= \frac{1}{B'^2+C'^2} \left[B'(b')^\alpha + B'\alpha T^{\alpha-1} - C'\alpha T^{*(\alpha-1)}\right]$$

$$T^{*(\alpha-1)} = \int_{b'}^\infty e^{-B'(R-b')} \sin C'(R-b')R^{\alpha-1} dR$$

$$T^o = \frac{-B'}{B'^2+C'^2}$$

$$T^{*o} = \frac{-C'}{B'^2+C'^2}$$

For moderate concentrations the expression of I^α using the virial expansion for g(R) is

$$I^\alpha = -(1+8w)\frac{b^{\alpha+1}}{\alpha+1} + 8w\left[\frac{(2b)^{\alpha+1}}{\alpha+1} + \frac{3b^{\alpha+1}(1-2^{\alpha+2})}{4(\alpha+2)} + \frac{b^{\alpha+1}}{16}\frac{(2^{\alpha+4}-1)}{(\alpha+4)}\right],$$

where b = 2a. For the lowest order of m in Eq. (15), which is useful in the low frequency scattering for monopole and dipole case, N_o^1 is obtained as

$$N_o^1 = 4\pi\rho I^2 \tag{17}$$
$$= -8w + 34w^2$$

which is the same result as Eq. (28) of Ref. 9.

For the low frequency scattering, an exact form of N_o^1 by using the Ornstein-Zerwike equation is

$$N_o^1 = 4\pi\rho \int_o^\infty [g(R)-1]R^2 dR = \frac{(1-w)^4}{(1+2w)^2} - 1 \tag{18}$$

which can be shown to be the same as Eq. (17) when the higher order terms $(O(w^3))$ are neglected after the expansion of Eq. (18).

APPROXIMATION IN THE LOW FREQUENCY RANGE

In the present study the suspended sedimentary particles are in general 2 to 4 microns in radius which make the nondimensional frequency ka in the range of 0.01 to 0.1 when using the ultrasonic wave of the frequency 1.2 to 12 megahertz. It is known that in the low frequency Rayleigh limit (ka<<1) only the monopole (n=o) and dipole (n=1) terms dominate the solution for the effective wave number K in Eq. (11). Thus Eq. (11) can be rewritten as

$$K^2 = k^2 + \frac{4\pi\rho}{k}(A_0 + A_1\eta^2) \tag{19}$$

or

$$\frac{K^2}{k^2} = \frac{1 + \frac{4\pi\rho}{k^3} A_0}{1 - \frac{4\pi\rho}{k^3} A_1}$$

where

$$A_0 = k\bar{f}_0 \; [1 + iA_0H_{00} + iA_1\eta H_{01}]$$

$$A_1 = k\bar{f}_1 \; [1 + iA_0\eta^{-1}H_{10} + iA_1H_{11}]$$

$$H_{00} = N_0^1 + iN_0{}^2$$

$$H_{01} = H_{10} = N_1^1 + iN_1{}^2$$

$$H_{11} = \frac{1}{3}\left[\frac{4\pi\rho}{k^3} i + H_{00}\right] + \frac{2}{3}\left[N_2^1 + iN_2{}^2\right]$$

In the low frequency range, A_0 and A_1 can be approximated by[9]

$$A_0 = \frac{\bar{a}_0}{1-i\bar{a}_0 W}$$

$$A_1 = \left(\frac{\bar{a}_1}{1 + \frac{4\pi\rho^3}{k^3}\ \frac{\bar{a}_1}{3} - \frac{\bar{a}_1}{3}i\ W}\right) \tag{20}$$

where

$$W \sim 1 + 4\pi\rho\int_0^\infty [g(R)-1]R^2 dR$$

$$\bar{a}_0 = \int_0^\infty \frac{i}{1+iC_0}\ q(a)da$$

$$\bar{a}_1 = \int_0^\infty \frac{3i}{1+iC_1}\ q(a)da$$

The limiting values of $\bar{a}_0$ and $\bar{a}_1$ for $k\bar{a}<<1$ are, respectively

$$\bar{a}_0 = Da_{0f}G(\bar{a})i \quad , \quad \text{fluid spheres}$$

$$\bar{a}_1 = -3Da_{1f}G(\bar{a})i$$

$$\bar{a}_0 = D(-1)G(\bar{a})i \quad , \quad \text{rigid spheres}$$

$$\bar{a}_1 = -3D\,\frac{-1}{2}\,G(\bar{a})i$$

$$\bar{a}_0 = Da_{0e}G(\bar{a})\,i \quad , \quad \text{elastic spheres}$$

$$\bar{a}_1 = -3Da_{1f}G(\bar{a})\,i$$

where

$D = (ka)^3/3$

$G(\bar{a})$ = the size distribution factor

$a_{0f} = 1/e - 1$

$a_{1f} = (1-g) / (1+2g)$

$$a_{0e} = \frac{1}{e-\frac{4}{3}\, e_1} - 1$$

$e = g(C_L/C_0)^2$

$e_1 = g(C_T/C_0)^2$

$\bar{a}$ = mean size (radius)

g = density ratio between the scatterer and the surrounding medium

C_0, C_L, C_T = wave speed in the surrounding medium, compressional wave speed in the scatterer, shear wave speed in the scatterer.

One sees from Eqs. (19) and (20) that the effective wave number K is a complex number and can be written as

$$K = K_R + iK_I, \; (K_I > 0).$$

In general, the real part of $K(K_R)$ is much greater than the imaginary part of $K(K_I)$ and Eq. (19) can thus be rewritten as, after separating the real and imaginary part of K,

$$\left(\frac{K_R}{k}\right)^2 \doteq \frac{(1+ws)(1-wt)}{1+2wt} \tag{21}$$

$$\frac{2K_I}{K_R} \doteq Dw(1-8w+34w^2)\left[\frac{s^2}{1+sw} + \frac{3t^2}{1+2wt(1-wt)}\right] \tag{22}$$

where

$$s = \begin{pmatrix} a_{0f} \\ -1 \\ a_{0e} \end{pmatrix} G(\bar{a}) \;, \quad \begin{matrix} \text{fluid} \\ \text{rigid} \\ \text{elastic} \end{matrix} \; \text{sphere}$$

$$t = \begin{pmatrix} a_{1f} \\ -\frac{1}{2} \\ a_{1f} \end{pmatrix} G(\bar{a}) \;, \quad \begin{matrix} \text{fluid} \\ \text{rigid} \\ \text{elastic} \end{matrix} \; \text{sphere}$$

For a uniform size distribution, i.e., $G(\bar{a})=1$, the results of (21) and (22) are identical to Eq. (73)[9] obtained by Twersky.

Based on Eqs. (21) and (22) the normalized phase velocity K_R/k and the loss tangent $2K_I/K_R$ are calculated for waves propagating through the water containing suspended particles, modelled as elastic spheres. When the concentration is higher than about 5% the term $(1-8w+34w^2)$ is replaced by $(1-w)^4/(1+2w)^2$ in Eq. (22) to correct for the higher concentration in calculating the loss tangent. The rigid scatterers are also used in the calculation for comparison purpose and in this case Eq. (11) becomes

$$K^2 = k^2 + \frac{w}{2}k^2 + \frac{i(ka)^3}{3} \cdot \frac{1}{4} \cdot \frac{w(1-w)^3}{(1+2w)^2}(7+2w)k^2 \tag{23}$$

which is Twersky's Eq. (75)[9] for uniform size scatterers. The size distribution factor is considered to be from the Gaussian size distribution and in this case is[1]

$$G(\bar{a}) = \frac{2}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\bar{a}}{m}\right)^2}\left(1+\frac{1}{2}\left(\frac{\bar{a}}{m}\right)^2\right)\left(\frac{m}{\bar{a}}\right)^3 + \frac{3}{2}\left(2-e^{-\frac{1}{\sqrt{2}}\left(\frac{\bar{a}}{m}\right)}\right)\left(\frac{m}{\bar{a}}\right)^2 + \frac{3}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\bar{a}}{m}\right)^2}\left(\frac{m}{\bar{a}}\right) + \frac{1}{2}\left(1+\mathrm{erf}\left(\frac{1\bar{a}}{\sqrt{2}m}\right)\right) \tag{24}$$

where m = the standard deviation.

RESULTS AND DISCUSSIONS

The acoustic wave propagating through a fluid medium containing a random distribution of suspended particulats with a considerable concentration is usually dispersed and attenuated. The effective wave number K for densely distributed scatterers can be shown to be different from that for sparsely distributed scatterers due to the pair-correlation between scatterers. The dispersion of the phase velocity is characterized by the real part of $K(K_R)$ and the attenuation of the wave intensity can be described by the imaginary part of $K(K_I)$. The effective wave number K is frequency dependent except at the low frequency range in which the normalized phase velocity K_R/k is independent of frequencies as shown in Eq. (21).

The normalized phase velocity K_R/k decreases as the concentration increases at the low frequency range which can be seen from Figure 4. The dispersion of the phase velocity has a dependence on the size distribution also. One sees from the plot that particles with a higher standard deviation in the Gaussian size distribution make the wave disperse faster. At the zero concentration there should be no dispersion. Therefore the ratio between the effective wave number and the incident wave number becomes one as verified by Eq. (21).

The dispersion of the wave velocity in a medium containing the uniform size scatterers is generally not so strong as that in a medium containing size distributed scatterers. One interesting phenomenon can be seen from Figure 4 that for the uniform size rigid spheres the normalized phase velocity instead of decreasi , increases with increasing concentration. This fact may be explained by the high impedance of the rigid body.

Figure 5 shows the relationship between the loss tangent $2K_I/K_R$ and the concentrations at $k\bar{a} = 0.1$, i.e. low frequency range. As can be seen from the figure, the attenuation increases with the increasing concentration until it reaches the maximum at the concentration of fifteen percent then decreases with the increasing concentration up to forty percent. As a matter of fact, this phenomenon and the saturation concentration of 15% is valid for all cases at the low frequency range and has also been verified by experiments.[13] For scatterers with the Gaussian size distributions, one sees that the more the particle size deviates from its mean radius, the larger the attenuation is. Therefore the attenuation of suspended particles with the uniform size distribution, which is the special case of zero standard deviation in Eq. (26), is always smaller than that of particles with the Gaussian size distribution. It can also be noted that the attenuation of fixed rigid spheres is larger than the present sedimentary particles which are modelled as elastic spheres. One sees from Eq. (20) that the shear wave of the elastic material may affect the attenuation to a certain degree. However, this effect does not appear in the present problem due to its small magnitude.

The loss tangent is frequency dependent and its dependence in the low frequency range for a fixed concentration (10%) is shown in Figure 6. On the log-log plot one sees that all straight lines have the slope three as predicted by the Rayleigh scattering theory. The attenuation increases with the increasing frequency in the Rayleigh region ($k\bar{a} \ll 1$). The scatterers with a larger standard deviation in the Gaussian size distribution will reach the assigned attenuation at a lower frequency. The dispersion and the attenuation of waves using Equation (7) based on the single scattering theory and Eq. (3) from Lloyd and Berry[13] are also shown in Figures 4 and 7 respectively. One sees that unless the concentration is very low the single scattering theory fails to predict the propagation characteristics.

Lloyd's equation predicts the wave dispersion well but fails in determining the attenuation. Eq. (4) using the hole correction is badly affected by the acoustic properties of scatterers. This can be seen from the attenuation for rigid and elastic scatterers (predicts better for rigid spheres). However, it is still not possible to obtain the right attenuation unless the concentration is very low. If attention is paid, one can see that there is no correction for the acoustic properties due to the concentration in Figs. (2) and (4) as it appears implicitly in Eq. (19)(finally appeared in Eqs. (21) and (22)) due to the consideration of the pair-correlation between scatterers. Besides the lack of the correction for a higher concentration. This may be the reason why the single scattering theory and its modified equations predict a lower dispersion and a much higher attenuation.

The hole correction equation representing the pair-correlation function is not suitable for the concentration greater than about five percent as can be seen from Figure 7. Besides this the hole correction equation will generate negative attenuation coefficients which are nonphysical results even starting at a comparatively low concentration.

When one is interested in the wave propagation through a higher concentration of scatterers in the high frequency range, an iterative solution can be obtained using a computer using suitable numerical methods. The proposed radial distribution function (Eq. 13)) is useful for pair-correlated elastic spheres whenever the concentration dependent parameters are decided. However, this will be left to the next paper.

TABLE I. Elastic Properties of the Sediments of the Abyssal Hill used in Calculation.
(Data from Stoll's[17] paper)

Range of Grain Sizes (a)	2 - 4 μm
Grain Density (ρ_E)	2.65 g/cm^3
Shear Wave Velocity (C_T)	210 m/sec
Compressional Wave Velocity (C_L)	3690 m/sec

water density 1 g/cm^3 and sound speed in water 1500 m/sec are used.

ACKNOWLEDGMENTS

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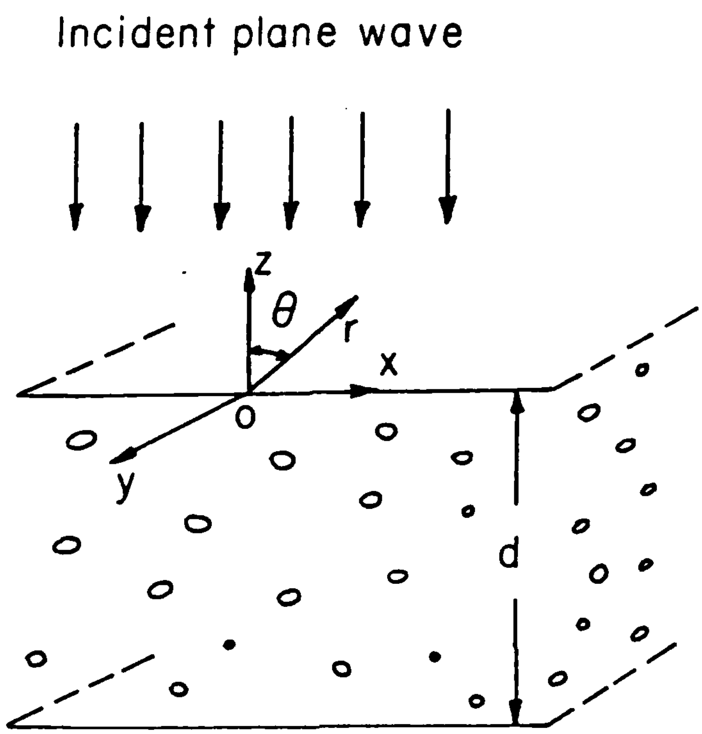

Figure 1. Geometry of the incident wave on a slab containing a distribution of scatterers of varying size

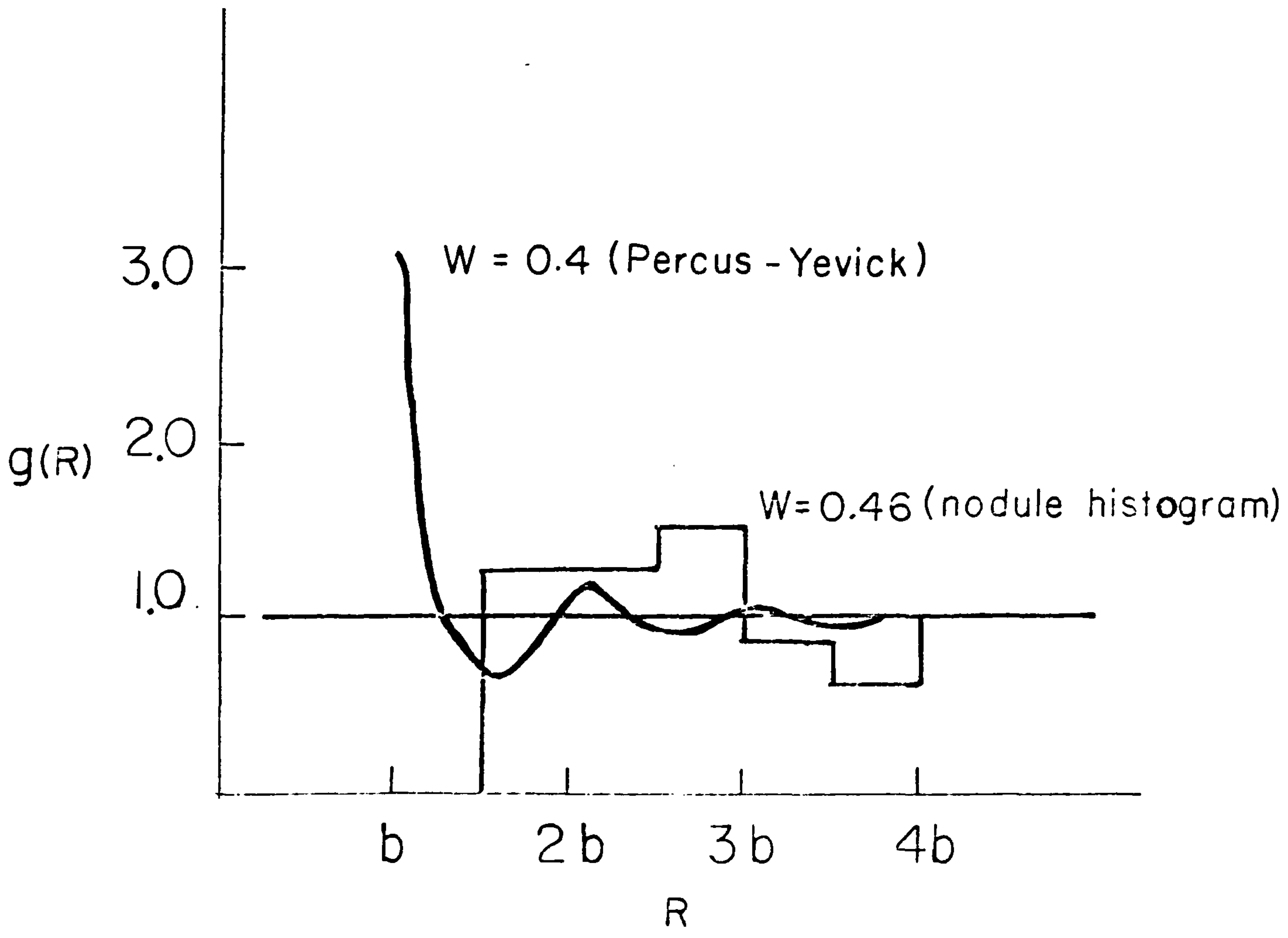

Figure 2. The radial distribution function g(R) from Percus-Yevick Equation and Nodule Histogram

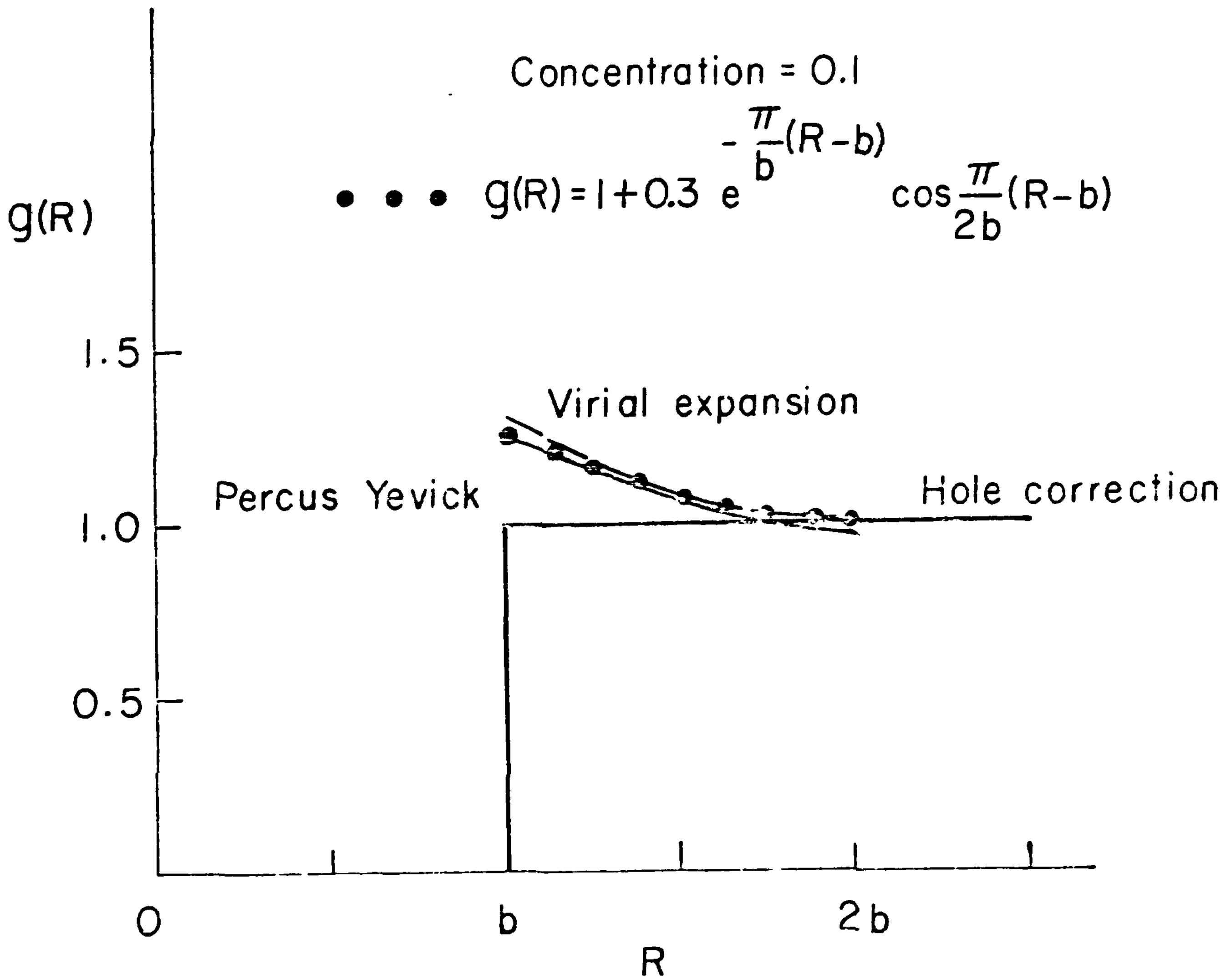

Figure 3. A comparison of g(R) using different expressions

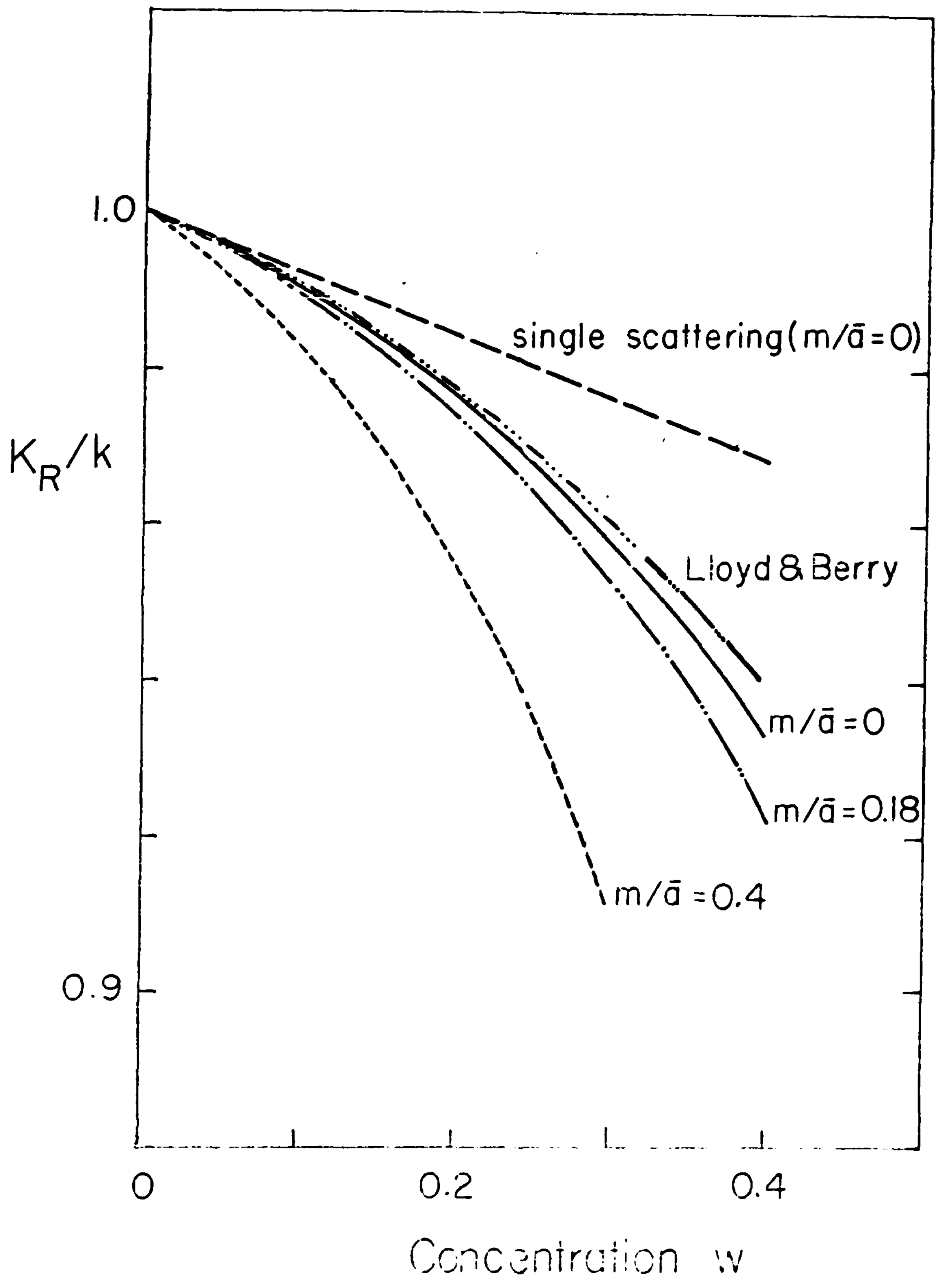

Figure 4. Phase velocity vs. w for Gaussian and Uniform Size Distributions

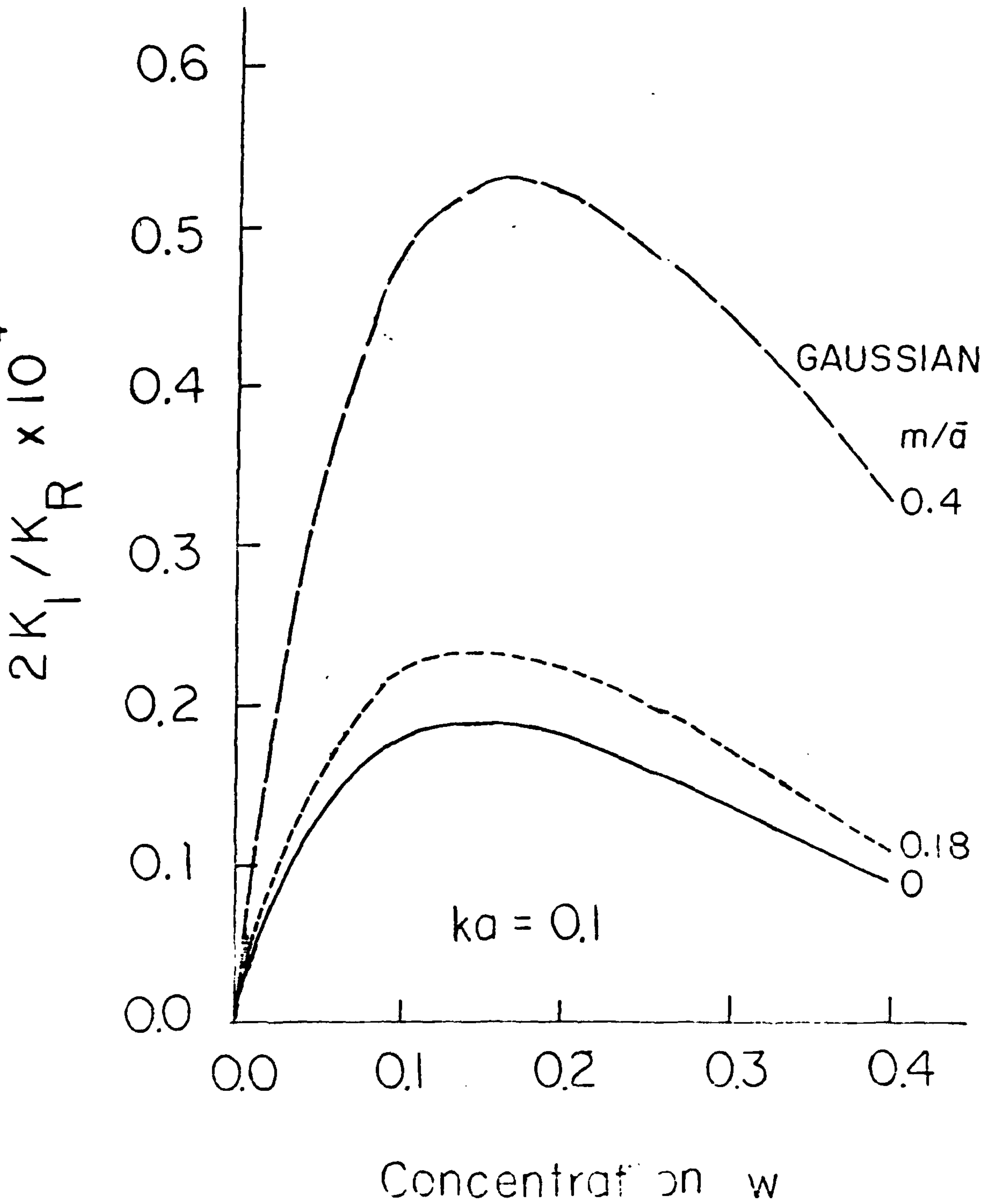

Figure 5. Coherent Attenuation vs. w for Gaussian and Uniform Size Distributions

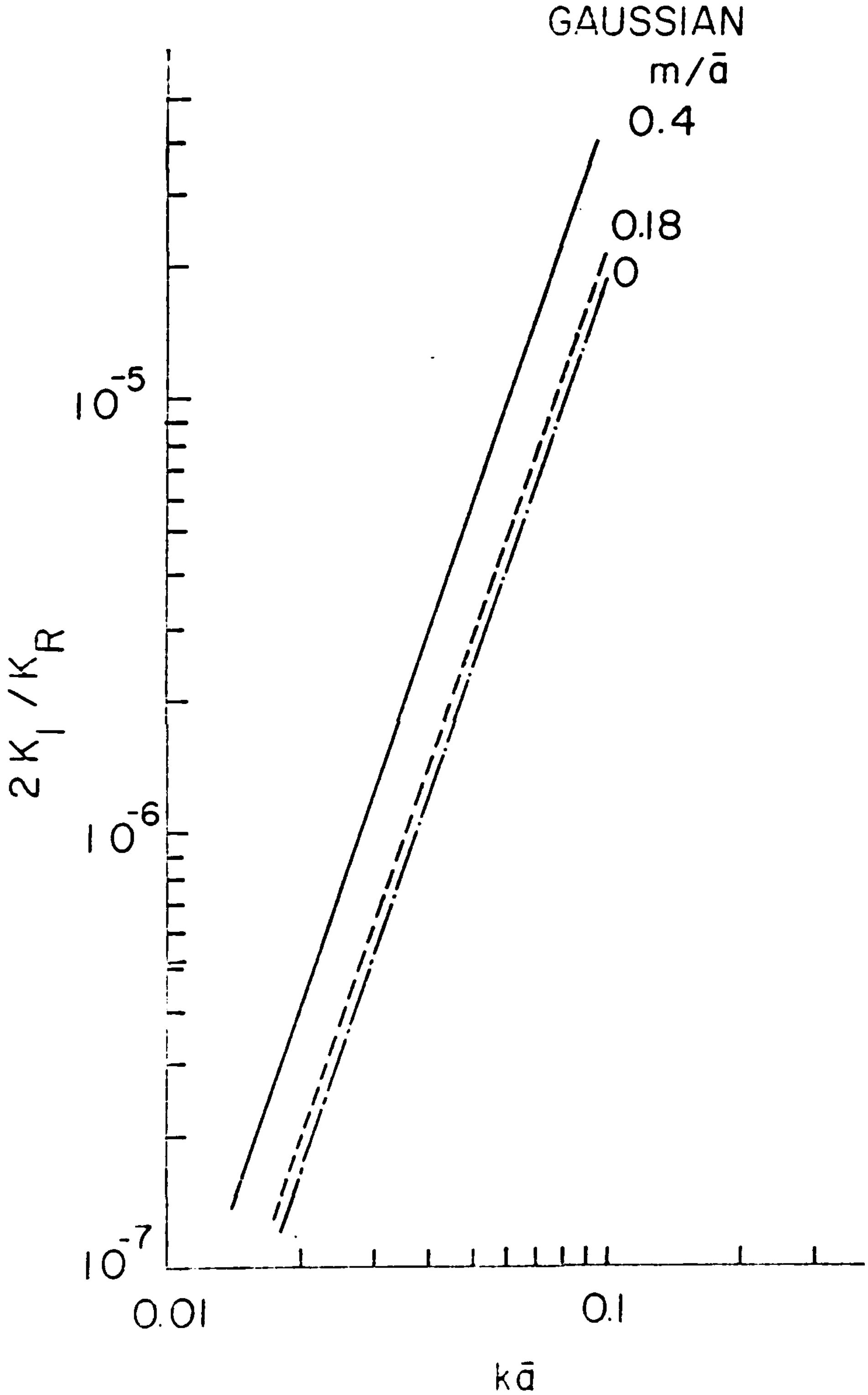

Figure 6. Coherent Attenuation vs. kā in Rayleigh Range

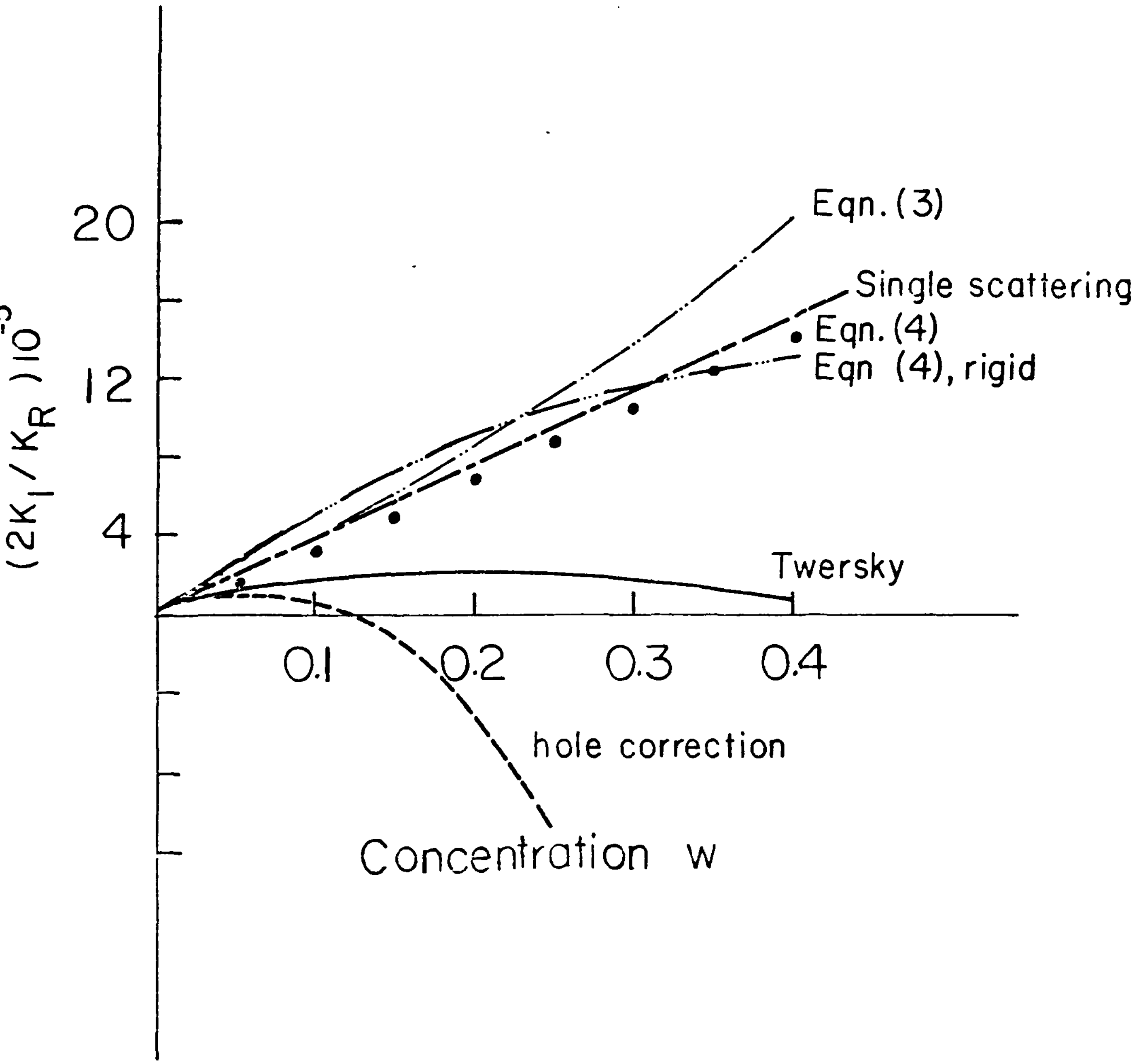

Figure 7. Coherent Attenuation vs. w -- Comparison of Various Theories

APPLICATION OF TWERSKY'S MULTIPLE SCATTERING FORMALISM TO A DENSE SUSPENSION OF ELASTIC PARTICLES IN WATER

by

Y. Ma*, V.K. Varadan* and V.V. Varadan*
Wave Propagation Group
Department of Engineering Mechanics
The Ohio State University, Columbus, Ohio 43210

ABSTRACT

Acoustic wave propagation through a dense suspension of solid elastic particles in water is studied. The particles in suspension have a size distribution and their relative positions are described by a pair-correlation function. Twersky's multiple scattering formalism is employed to obtain new analytical expressions for the phase velocity and coherent attenuation of a wide range of concentrations. Numerical results presented are of interest in the study of marine sediments.

INTRODUCTION

Recent advances in the area of multiple scattering in discrete random media by Twersky[1-3] have shown that his technique can uniformly handle scattering by acoustic and electromagnetic waves. He has obtained solutions in the long-wavelength limit for the bulk propagation constants by treating the particles as "hard" spheres in a statistical-mechanical sense. In this paper, his approach is applied to study acoustic wave propagation in a fluid containing a dense suspension of spherical elastic particles of varying sizes. Closed form expressions are obtained for both phase velocity and coherent attenuation in the Rayleigh limit. Numerical results are presented at higher frequencies for marine sediments for various values of concentrations.

SOLUTION FOR RADIALLY SYMMETRIC SCATTERERS

Consider a random distribution of elastic spheres in water. The elastic properties of the scatterers are given by Lame's constants λ_1 and μ_1 and density ρ_1. The fluid properties are given by compressibility λ and density ρ. The longitudinal and shear wave velocities in the elastic scatterer are given by $c_p = [(\lambda_1 + 2\mu_1)/\rho_1]^{1/2}$ and $c_s = (\mu_1/\rho_1)^{1/2}$ while the compressional wave velocity in water is given by $c = (\lambda/\rho)^{1/2}$.

The effective wavenumber K for a random distribution of pair-correlated scatterers can be written in terms of wavenumber $k(= \omega/c)$ in water as follows[1]:

$$K^2 = k^2 + \frac{4\pi n_o}{k} \sum_{n=o}^{\infty} A_n \eta^{2n} \tag{1}$$

where

$$\eta = K/k$$

$$A_n = k\, \overline{f_n(a,0)}[1 + \sum_{m=o}^{\infty} iA_n(n)^{-n+m} H_{nm}] \tag{2}$$

where

$$\overline{f_n(a,0)} = \int_o^\infty f_n(a,0)\ q(a)da \quad ; \quad f_n(a,0) = \frac{i(2n+1)}{1+iC_n}$$

C_n is a complex function containing density, compressibility and rigidity of the elastic scatterer and different orders of spherical Bessel and Neumann functions, see Appendix A and

$$H_{nm} = \sum_{|n-m|}^{n+m} d_m\begin{pmatrix} 0 & ; & 0 \\ n & & m \end{pmatrix} \left[\frac{i4\pi n_o}{k^3} \frac{\eta^{m+n}-\eta^m}{\eta^2-1} + N_m \right] \tag{3}$$

$$N_m = N_m^1 + iN_m^2 = 4\pi n_o \int_o^\infty [g(R)-1] j_m(KR) h_m^{(2)}(kR) R^2 dR$$

In Eqs. (2) and (3), $n_o = N/V$ is the number density (number of scatterers per unit volume), $f_n(a,0)$ is the forward scattering amplitude of a spherical scatterer of radius 'a', $q(a)$ is the size distribution function, $g(R)$ is the pair-correlation functions, $j_m(\ )$ and $h_m^{(2)}(\ )$ are the spherical Bessel and Hankel functions, respectively, and the coefficients $d_m\begin{pmatrix} 0 & ; & 0 \\ n & & m \end{pmatrix}$ can be obtained from the expansion of the Legendre polynomials as given by

$$P_n(x)\ P_m(x) = \sum_{q=0}^{n+m} d_q\begin{pmatrix} 0 & ; & 0 \\ n & & m \end{pmatrix} P_q(x)\ . \tag{4}$$

$d_q\begin{pmatrix} 0 & ; & 0 \\ n & & m \end{pmatrix}$ has the following properties:

$$d_q\begin{pmatrix} 0 & ; & 0 \\ n & & m \end{pmatrix} = d_q\begin{pmatrix} 0 & ; & 0 \\ m & & n \end{pmatrix}$$

$$d_o\begin{pmatrix} 0 & ; & 0 \\ n & & m \end{pmatrix} + d_1\begin{pmatrix} 0 & ; & 0 \\ n & & m \end{pmatrix} +\ldots+ d_q\begin{pmatrix} 0 & ; & 0 \\ n & & m \end{pmatrix} = 1\ ;\ q = m+n \tag{5}$$

$$d_{odd}\begin{pmatrix} 0 & ; & 0 \\ m & & n \end{pmatrix} = 0 \quad ; \quad m+n = \text{even}$$

$$d_{even}\begin{pmatrix} 0 & ; & 0 \\ m & & n \end{pmatrix} = 0 \quad ; \quad m+n = \text{odd}$$

The effective wavenumber $K(= K_1+iK_2)$ is complex, the real and imaginary parts of which are related to the phase velocity and coherent attenuation, respectively.

For monopole and dipole cases

$$d_o\begin{pmatrix}0 & ; & 0\\ 0 & & 1\end{pmatrix} = 0 \quad , \quad d_1\begin{pmatrix}0 & ; & 0\\ 0 & & 1\end{pmatrix} = 1 \tag{6}$$

In the Rayleigh limit, only monopole and dipole terms and a two-term expansion of (1) are considered in obtaining a closed form expression for the phase velocity and attenuation. For higher frequencies, the general form of d_q and more terms in expansion of Eq. (1) should be employed. This is best done numerically.

To calculate the effective wavenumber K, the equation (3) containing N_m which involves the radial distribution function g(R) needs to be obtained first. For a more precise calculation, the correlation integral, Eq. (3), should be computed for each value of **a**. However, we have chosen to compute the correlation integral for the mean sphere. This is discussed in more detail in what follows. Following the work by Twersky, we can write

$$N_m = 4\pi n_o\, F(K,k)\, I^{\alpha} \quad ; \quad I^{\alpha} = \int_0^{\infty} [g(R)-1]R^{\alpha}dR \tag{7}$$

where F(K,k) is a function of the effective wavenumber K and the host medium wavenumber k. Using the Virial expansion[1] for g(R), we can show that

$$I^{\alpha} = -(1+8w)\,\frac{b^{\alpha+1}}{\alpha+1} + 8w \quad \frac{(2b)^{\alpha+1}}{\alpha+1} + \frac{3b^{\alpha+1}(1-2^{\alpha+2})}{4(\alpha+2)} + \frac{b^{\alpha+1}(2^{\alpha+4}-1)}{16(\alpha+4)} \tag{8}$$

where $b = 2\bar{a}$ and $w = n_o 4\pi\bar{a}^3/3$ is the concentration of scatterers and $\bar{a} = \sqrt[3]{4\pi N/3w}$ is the mean radius. For the lowest order of m in (7) which is useful for low frequency scattering, N_o^1 is obtained as

$$N_o^1 = 4\pi n_o I^2 = -\, 8w + 34w^2 \tag{9}$$

The Virial series expansion and the resulting Eq. (9) seem to be good for low concentrations $w<0.05$. For higher concentrations, one can obtain the following expression derived by Twersky for N_o^1 by using Ornstein-Zernicke equation given by

$$N_o^1 = \frac{(1-w)^4}{(1+2w)^2} - 1 \tag{10}$$

Eq. (10) is preferred to Eq. (9) since it is not restricted to small values of w. But corresponding expressions cannot be obtained for all values of α, hence we have to resort to the virial expansion.

PHASE VELOCITY AND COHERENT ATTENUATION

The suspended sedimentary particles in water are in general 2 to 4 microns in radius. For an acoustic wave with a frequency range of 1.2 to 12 MHz, this corresponds to a non-dimensional wavenumber ka in the range of 0.01 to 0.1. For such long wavelength acoustic wave propagation studies, it is enough to keep only monopole and dipole terms in Eq. (1). The resulting dispersion equation is given by

$$\frac{K^2}{k^2} = \frac{1 + \frac{4\pi n_o}{k^3} A_o}{1 - \frac{4\pi n_o}{k^3} A_1} \tag{11}$$

where

$$\begin{aligned}
A_o &= kf_o[1 + iA_oH_{oo} + iA_1\eta H_{o1}] \\
A_1 &= kf_1[1 + iA_o\eta^{-1}H_{1o} + iA_1H_{11}] \\
H_{00} &= N_0^1 + iN_0^2 \\
H_{01} &= H_{10} = N_1^1 + iN_1^2 \\
H_{11} &= \frac{1}{3}[i\frac{4\pi n_o}{k^3} + H_{00}] + \frac{2}{3}[N_2^1 + iN_2^2]
\end{aligned} \tag{12}$$

Following Twersky's work[1], A_o and A_1 can be written as follows

$$A_o = \frac{\bar{a}_o}{1-i\bar{a}_oW}$$

$$A_1 = \frac{\bar{a}_1}{1 + \frac{4\pi n_o}{k^3}\frac{\bar{a}_1}{3} - i\frac{\bar{a}_1}{3} W} \tag{13}$$

where

$$W \approx 1 + 4\pi n_o \int_o^\infty [g(R)-1]R^2 dR$$

$$\bar{a}_o = \int_o^\infty \frac{i}{1+iC_0} q(a)da$$

$$\bar{a}_1 = \int_o^\infty \frac{3i}{1+iC_1} q(a)da \tag{14}$$

For $k\bar{a} << 1$, we obtain the following limiting values for $\bar{a}_o$ and $\bar{a}_1$

$$\bar{a}_0 = \frac{i(ka)^3}{3} a_{0e}\, G(\bar{a})$$

$$\bar{a}_1 = -i(ka)^3 a_{1f}\, G(\bar{a}) \tag{15}$$

where

$$a_{0e} = \frac{1}{g[(\frac{c_p}{c})^2 - \frac{4}{3}(\frac{c_s}{c})^2]} - 1$$

$$a_{1f} = \frac{1-g}{1+2g} \tag{16}$$

In Eqs. (15) and (16), g is the density ratio between the scatterer and the surrounding medium and c_p, c_s and c are the longitudinal and transverse wave speeds of the elastic scatterer and the compressional wave speed of the surrounding fluid. As a limiting case, we can also obtain the dispersion relations for a random distribution of either rigid ($\lambda_1, \mu_1 \to \infty$) or fluid ($\mu_1 \to 0$) scatterers in water.

After some manipulations, one can obtain the real (K_1) and imaginary part (K_2) of the effective wavenumber K as follows

$$\left(\frac{K_1}{k}\right)^2 = \frac{(1+w\alpha)(1-w\beta)}{1+2w\beta} \tag{17}$$

$$\frac{2K_2}{K_1} = \frac{1}{3}(ka)^3 w\gamma\left[\frac{\alpha^2}{1+\alpha w} + \frac{3\beta^2}{1+2w\beta(1-w\beta)}\right]$$

where

$$\alpha = \begin{Bmatrix} a_{0f} \\ -1 \\ a_{0e} \end{Bmatrix} G(\bar{a}) \quad , \quad \begin{matrix} \text{fluid} \\ \text{rigid} \\ \text{elastic} \end{matrix} \quad \text{spheres} \tag{18}$$

$$\beta = \begin{Bmatrix} a_{1f} \\ -1/2 \\ a_{1f} \end{Bmatrix} G(\bar{a}) \quad , \quad \begin{matrix} \text{fluid} \\ \text{rigid} \\ \text{elastic} \end{matrix} \quad \text{spheres} \tag{19}$$

In Eq. (17), $\gamma = (1-8w+34w^2)$ for $0<w<0.05$ and $\gamma=(1-w)^4/(1+2w)^2$ for $w>0.05$. For a uniform size distribution, $G(\bar{a})= 1$ while for a Gaussian size distribution

$$G(\bar{a}) = \sqrt{\frac{2}{\pi}} \exp\left[-\frac{1}{2}\left(\frac{\bar{a}}{m}\right)^2\right]\left[1 + \frac{1}{2}\left(\frac{\bar{a}}{m}\right)^2\right]\left(\frac{m}{\bar{a}}\right)^3$$

$$+ \frac{3}{2}\left[2 - \exp\left(-\frac{1}{\sqrt{2}}\left(\frac{\bar{a}}{m}\right)\right]\left(\frac{m}{\bar{a}}\right)^2$$

$$+ \frac{3}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\bar{a}}{m}\right)^2\right]\left(\frac{m}{\bar{a}}\right) + \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{\bar{a}}{\sqrt{2}m}\right)\right] \tag{20}$$

where m is the standard deviation.

The Gaussian distribution function has an infinite tail, however in the numerical calculations the cut off size was chosen to be $2\bar{a}$ where $\bar{a}$ is given by

$$\bar{a} = \sqrt[3]{4\pi N/3w}$$

With this cut off, for $\frac{m}{\bar{a}} = 0.4$, $q(2\bar{a}) = 4.4\times10^{-2}$ which is small (4% of peak value) and if $k\bar{a}$ is small $2k\bar{a}$ is still quite small and the long wavelength approximation is still valid for $m/\bar{a}$=0.18, $q(2\bar{a})=1.9\times10^{-7}$, which is negligible. The question may be raised as to the radius of the excluded spherical volume surrounding each scatterer. If we use $4\bar{a}$, the diameter of the largest size sphere of the distribution, although we would be correct in the integration on the allowed position for each scatterer, we would be excluding an unreasonable volume, thus limiting the volume fraction of scatterers. It seems more reasonable to take the diameter of the mean sphere, $2\bar{a}$ as the radius of the excluded spherical volume. On the average, this would be applicable to most scatterers and allow us to consider higher volume fractions. If the volume fraction is low enough, whether the radius of the excluded sphere is $2\bar{a}$ or $4\bar{a}$ will not pose a problem. For higher concentrations, if $m/\bar{a} < 0.5$, the procedure we have followed should yield good results.

Some numerical calculations were carried out for sediment particles in water. The particles are assumed to be spherical in shape and their elastic properties are given in Table I. In the Rayleigh limit, the normalized phase velocity K_1/k is frequency independent. Its dependence on concentration is shown in Fig. 1 for both uniform and Gaussian size distributions. The coherent attenuation (loss tangent) K_2/K_1, however, depends on frequency as depicted by Fig. 2. From the log-log plot of Fig. 2, one can see that the loss tangent for both uniform and gaussian size distributions is a straight line with a slope equal to 3 as predicted by Eq. (17). Figure 3 presents the loss tangent as a function of concentration at ka=0.1. The attenuation increases with increasing concentration until it reaches a maximum at c=15% and then decreases with increasing concentration. This kind of behavior has been observed experimentally by Lloyd and Berry[4].

For higher frequencies and concentrations, more terms must be kept in Eq. (1) along with suitable radial distribution function g(R). Our model of the random system is that the spherical scatterers cannot interpenetrate. In the statistical mechanics literature, this is synonymous with the ensemble of "hard" spheres. Several theories and calculations are available for determining the joint probability distribution function, viz., the Hypernetted-Chain Equation(HNC), the Percus-Yevick Approximation (P-YA), the Self-Consistent Approximation (SCA), Monte Carlo calculations, etc. We found the SCA is better suited for higher concentrations, see Ref. 5.

In Fig. 4, the normalized phase velocity is plotted as a function of frequency. The phase velocity displays an anomalous dispersion for $0.7<ka<1.0$. This anomaly is a result of some resonance effect associated with the individual particles. In Fig. 5, the attenuation is plotted as a function of frequency for the same concentration, $c=0.105$. The attenuation is a maximum in the frequency range where the phase velocity displays an anomalous behavior. However, since the calculations were made only up to $ka=1.0$, we cannot make a definitive statement in this regard. In Figs. 6 and 7, the normalized phase velocity and attenuation are plotted as a function of concentration at $ka=1.0$.

CONCLUSIONS

In this paper, Twersky's multiple scattering formalism has been extended to elastic scatterers in water. It has been shown that in addition to providing closed form expressions for the dispersion relations for elastic scatterers in the low frequency limit, his formalism is also well suited for numerical computations at higher frequencies and concentrations employing suitable pair-correlation functions.

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TABLE I. Elastic Properties of Sediments Used in Calculations[6]

Range of grain sizes (a)	2-4 μm
Grain density (ρ_1)	2.65 g/cm^3
Shear wave velocity (c_s)	210 m/sec
Compressional wave velocity (c_p)	3690 m/sec
Sound speed in water	1500 m/sec

APPENDIX A: THE EXPRESSION OF C_n USED IN EQ. (1)

$$C_n = \frac{n_{n1} D - g \frac{x_1}{x_3^2} n'_{n1} E}{-j_{n1} D + g \frac{x_1}{x_3^2} j'_{n1} E}$$

where

$$D = 2n(n+1) \left[1 - \frac{j_{n2}}{x_2 j'_n} \right] - \frac{x_3^2 j''_{n3}}{j_{n3}} - (n^2+n-2)$$

$$E = 4n(n+1) \left[1 - \frac{j_{n2}}{x_2 j'_{n2}} \right] \left[1 - \frac{x_3 j'_{n3}}{j_{n3}} \right]$$

$$-2x_2 \left\{ \left[\frac{x_3^2 j''_{n3}}{j_{n3}} + n^2 + n - 2 \right] \left[\left[\frac{1}{2h_3^2} - 1 \right] \frac{j_{n2}}{j'_{n2}} - \frac{j''_{n2}}{j'_{n2}} \right] \right\}$$

$n_{n1} = n_n(ka)$, $n_{n2} = n_n(k_p a)$

$j_{n1} = j_n(ka)$, $j_{n2} = j_n(k_p a)$, $j_{n3} = j_n(k_s a)$

$x_1 = ka$, $x_2 = k_p a$, $x_3 = k_s a$

$k_L = \omega / c_p$, $k_T = \omega/c_s$, $k = \omega/c$

$$j'_{n_{\substack{1\\2\\3}}} = \left.\frac{d\,j_n(k_{\substack{k\\p\\s}} r)}{d\,(k_{\substack{k\\p\\s}} r)}\right|_{r=a}$$

$$j''_{n_{\substack{2\\3}}} = \left.\frac{d^2 j_n(k_{\substack{p\\s}} r)}{d(k^2_{\substack{p\\s}} r^2)}\right|_{r=a}$$

$$n'_{n1} = \left.\frac{dn_n(kr)}{d(kr)}\right|_{r=a}$$

$$h_3 = C_T/C_L$$

n_n =the n-th order spherical Newman function, j_n =the n-th order spherical Bessel function.

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FIGURE CAPTIONS

Figure 1. Phase velocity vs concentration w in the Rayleigh limit for Gaussian ($m/\bar{a}$ = 0.18, 0.4) and uniform size ($m/\bar{a}$ = 0.0) distributions

Figure 2. Coherent wave attenuation vs ka for w=0.1 and $m/\bar{a}$ = 0.0, 0.18, 0.4

Figure 3. Coherent wave attenuation vs concentration w for ka = 0.1 and $m/\bar{a}$ = 0.0, 0.18, 0.4.

Figure 4. Normalized phase velocity vs ka for w=0.105.

Figure 5. Coherent wave attenuation vs ka for w=0.105.

Figure 6. Normalized phase velocity vs w for ka = 1.0.

Figure 7. Coherent wave attenuation vs w for ka = 1.0.

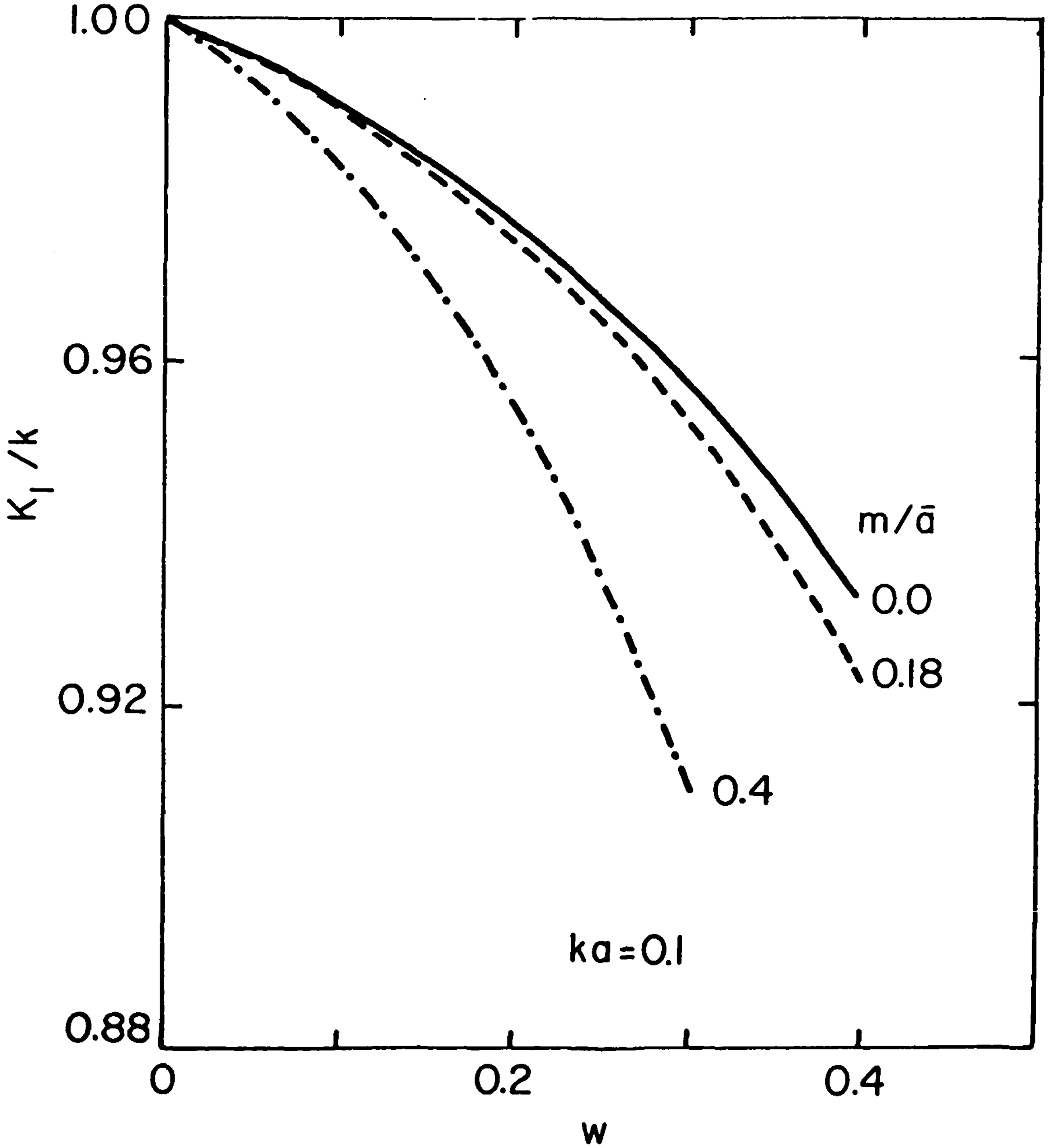
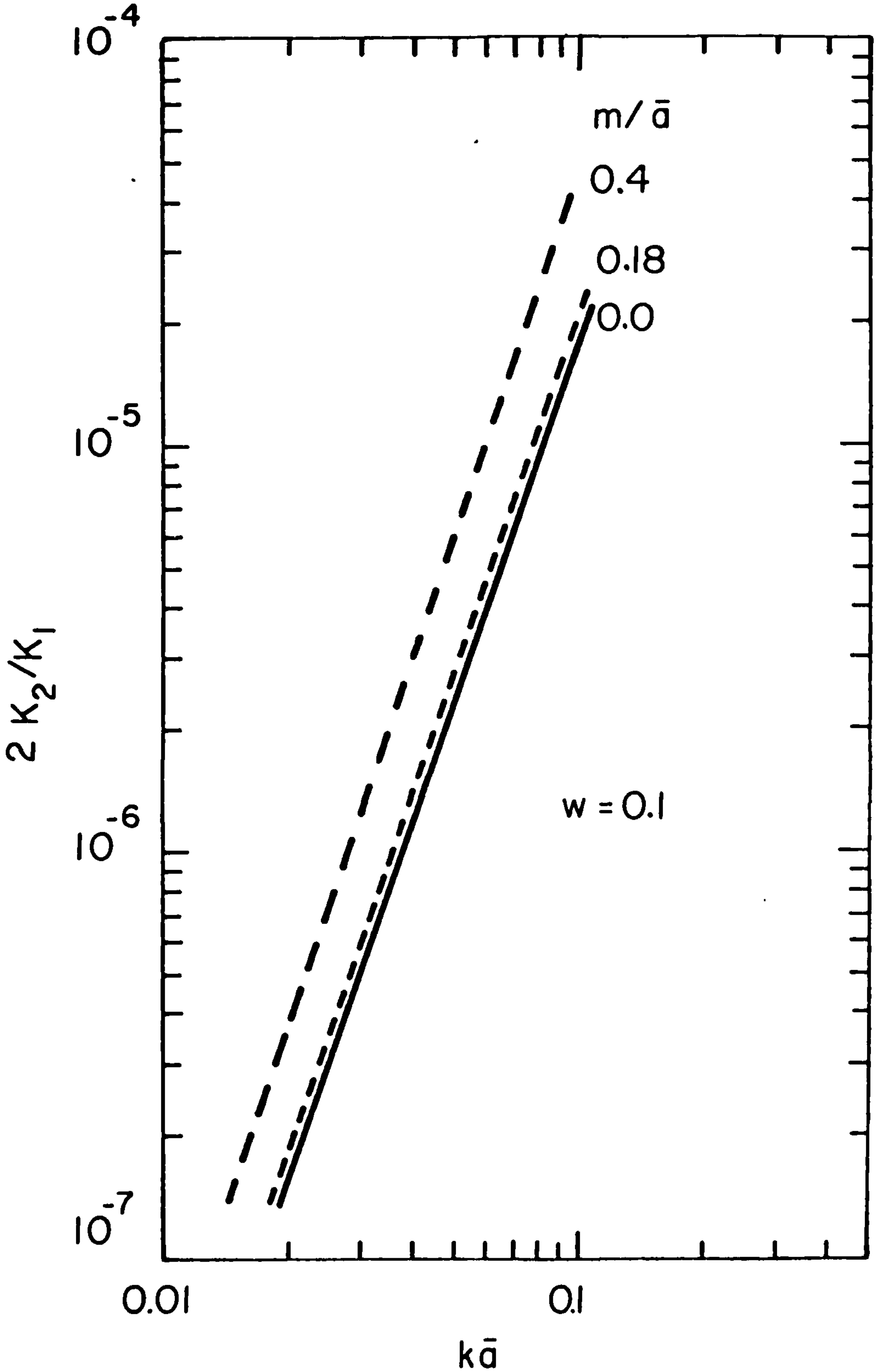


Figure 1



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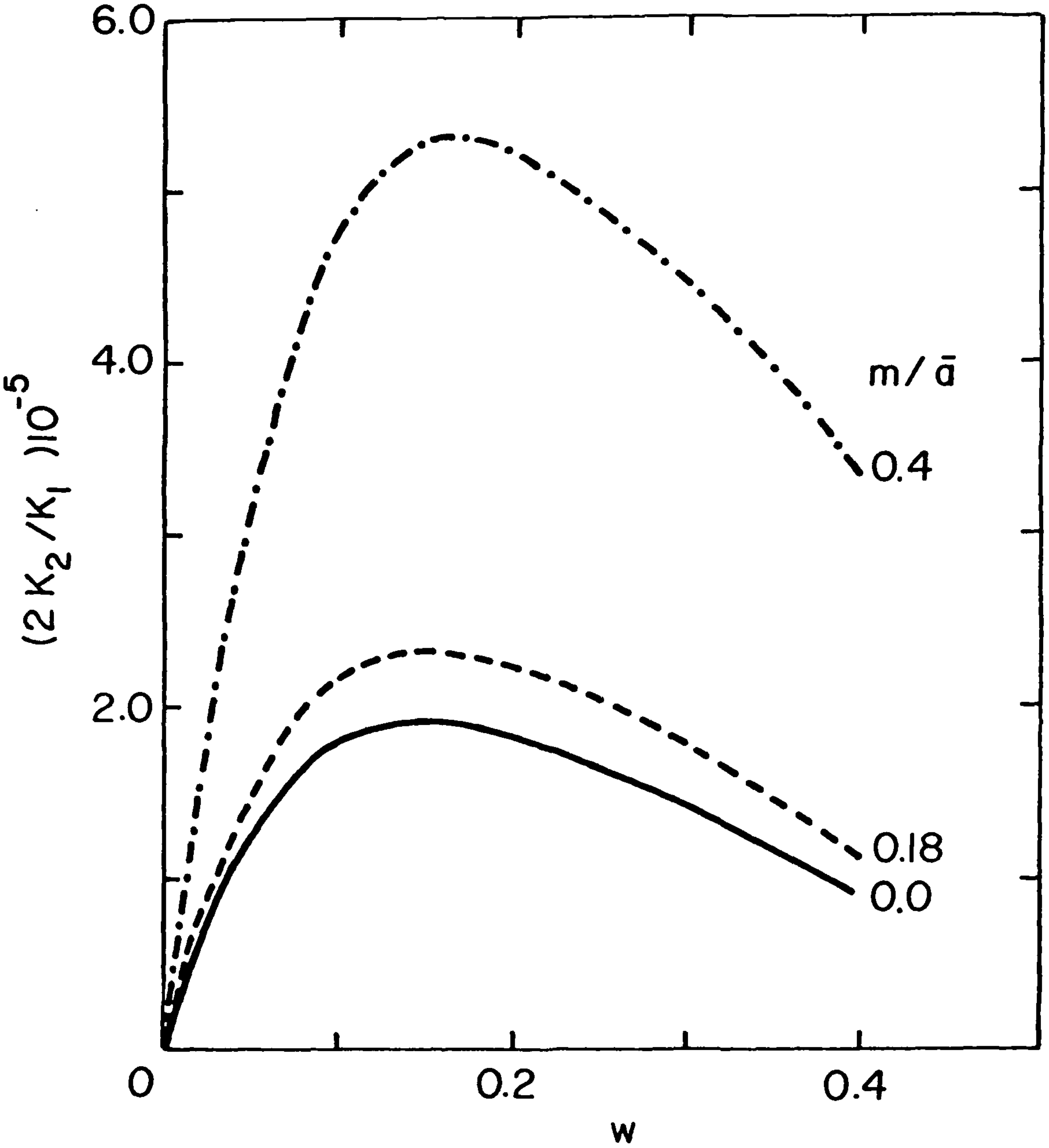

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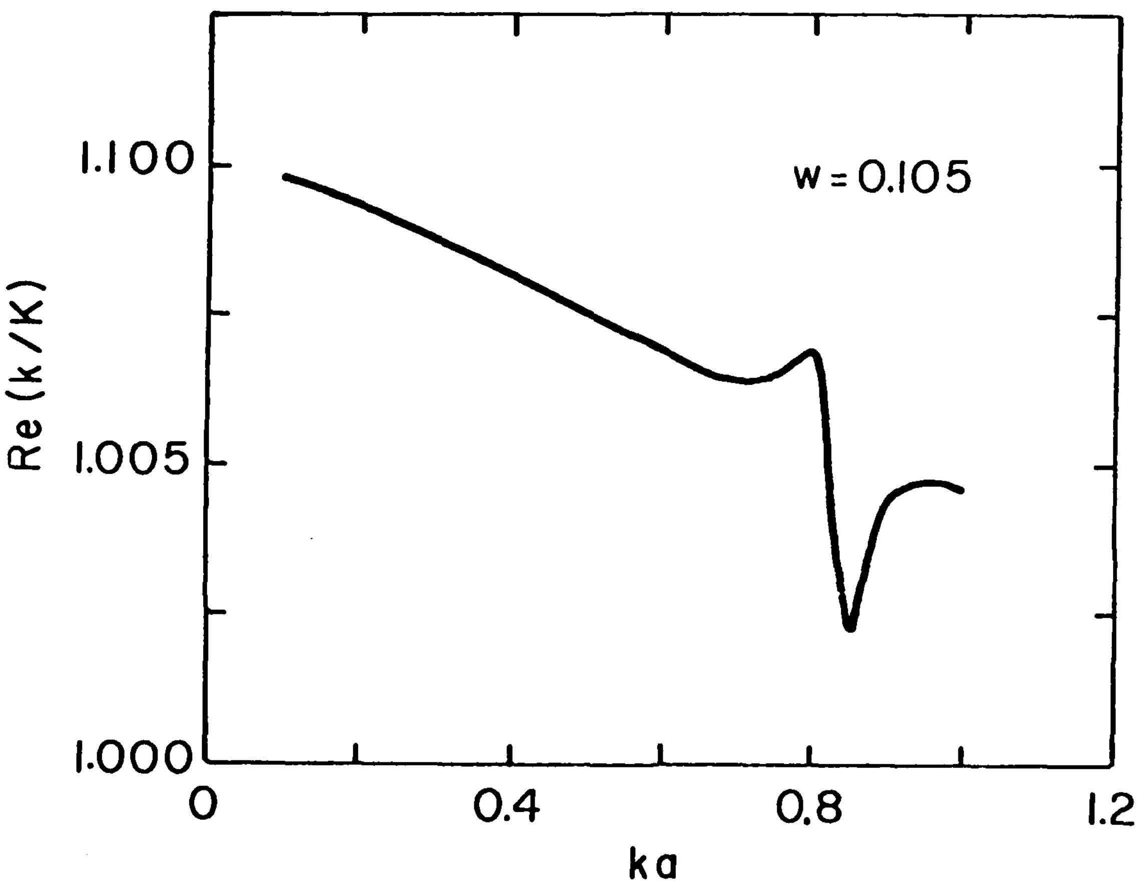

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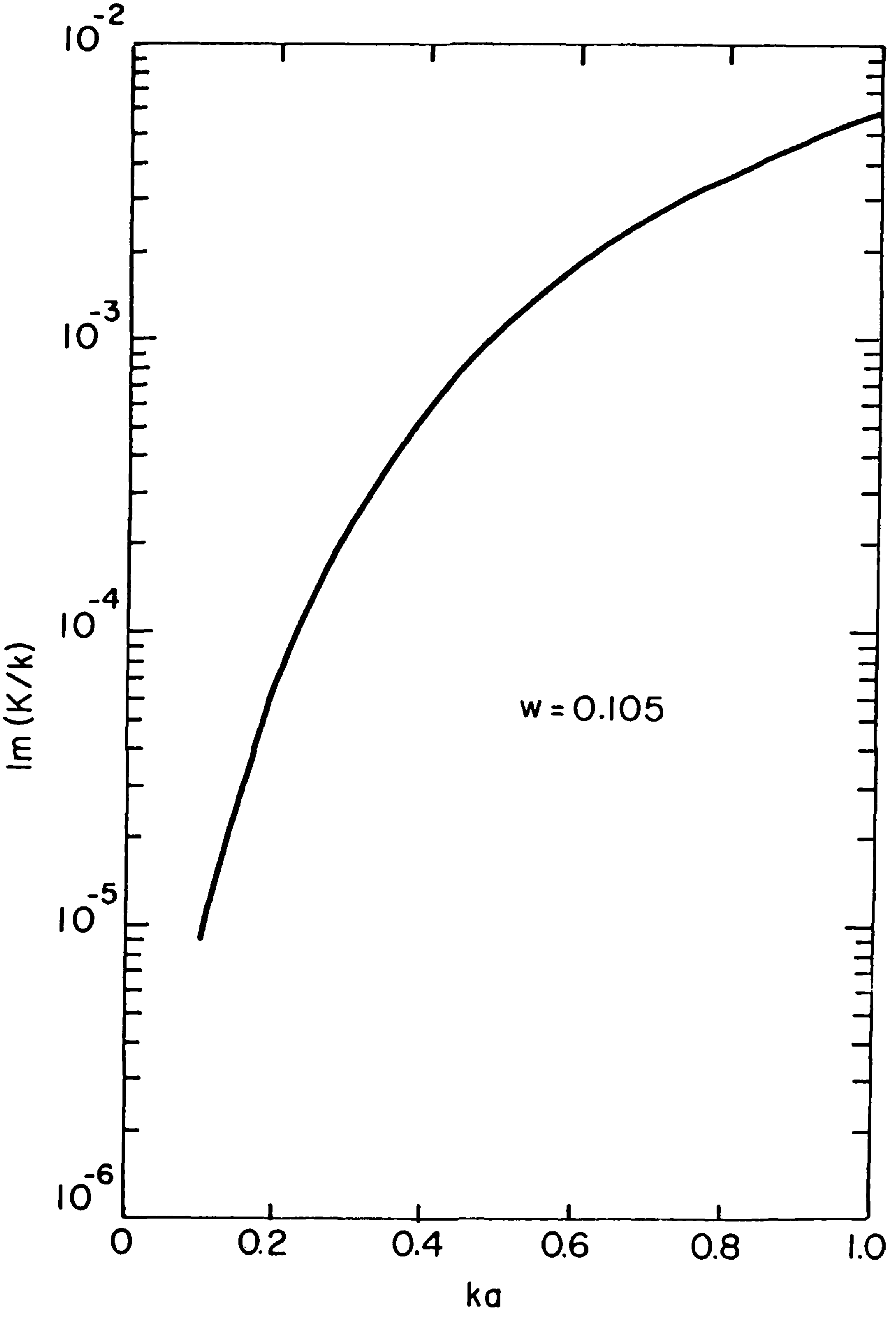

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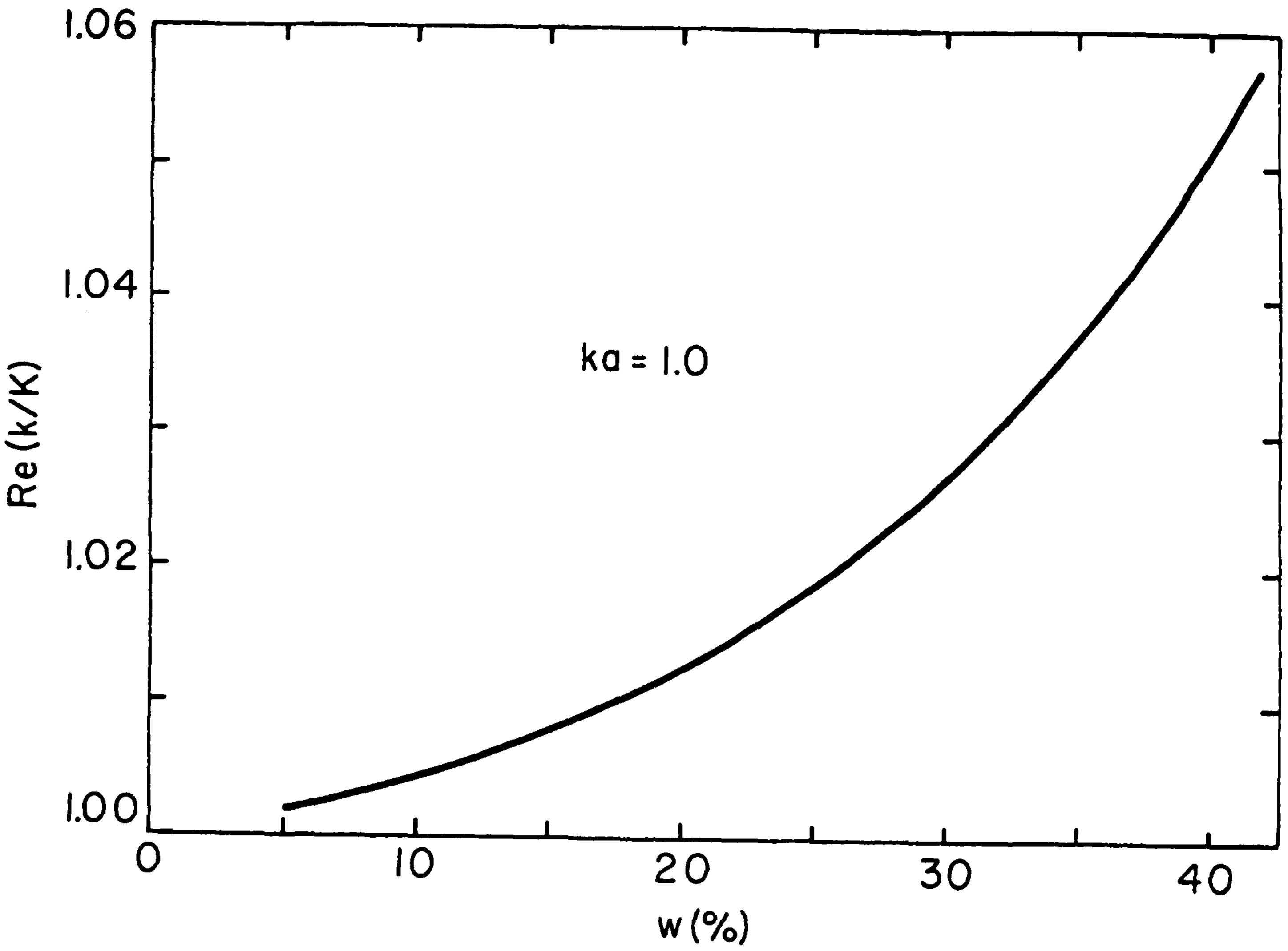

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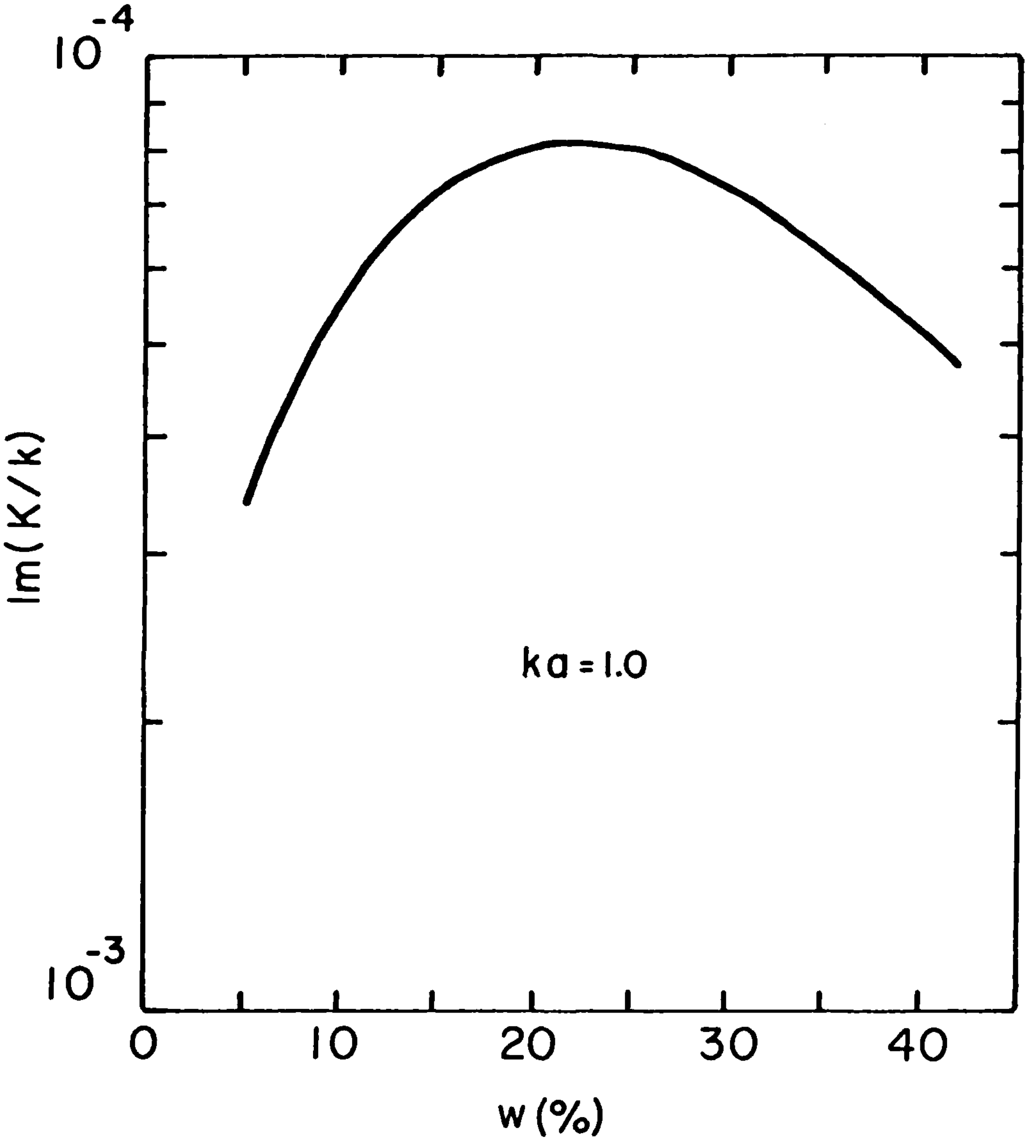

Figure 7

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The effect of multiple scattering on the coherent wave propagation is discrete random media is studied for spherical and non-spherical dielectric scatterers as a function of frequency and volume concentration of scatterers. The first and second order probability distribution functions are specified and a self-consitent T-matrix approach together with Lax's quasi-crystalline

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) approximation is used to derive dispersion equations whose singular solutions yield the complex propagation constant of the "effective medium." Various forms of pair-correlations were considered in the analysis; the self-consistent pair-correlation function is found to be well suited for a wide range of concentration of scatterers. The formalism is also found to be applicable for very high frequencies (ka>50). The theoretical results obtained by this formalism were compared with experimental findings and the agreement is excellent.

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